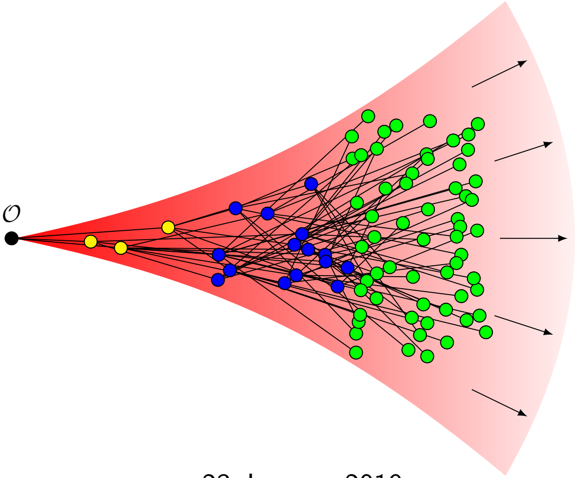
A Universal Operator Growth Hypothesis



Daniel PARKER

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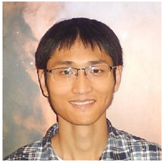
23 January 2019

Acknowledgements

Collaborators



Xiangyu Cao

Thomas Scaffidi

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Funding

European Research Council
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Advisor

Joel Moore

Operator Growth

- Start with a spin chain
- e.g. Chaotic Ising Model:

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i$$

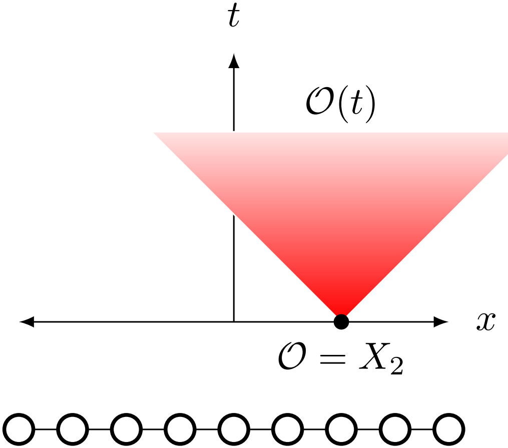
- Local Operator:

$$\text{e.g. } \mathcal{O} = X_2$$

- Unitary evolution $\mathcal{O}(t) = e^{-iHt}\mathcal{O}e^{iHt}$.
- Probe with correlation functions:

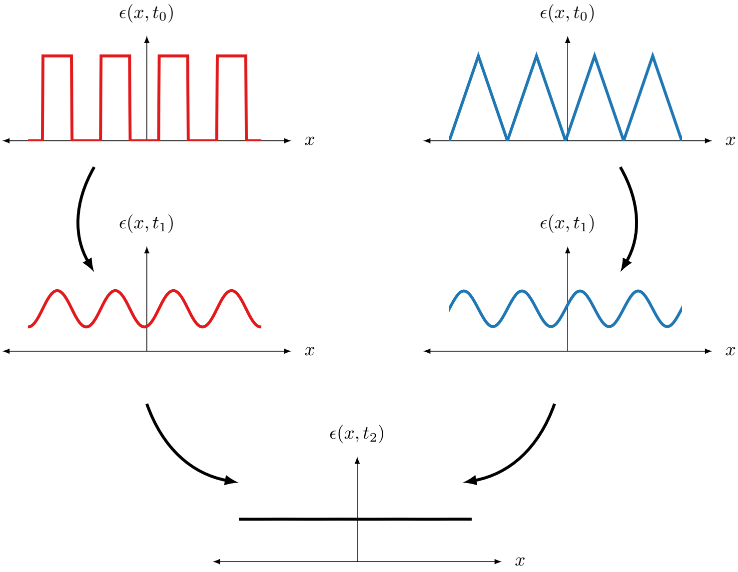
$$C(t) = \mathrm{Tr}[\mathcal{O}(t)\mathcal{O}(0)].$$

- **Exact, reversible** dynamics, but hard to compute.



Hydrodynamics

- Hydrodynamic descriptions valid at large time and wavelength
- Usually (classical) partial differential equations
- e.g. energy diffusion, spin diffusion, electron hydrodynamics
- Usually **irreversible or dissipative** dynamics.



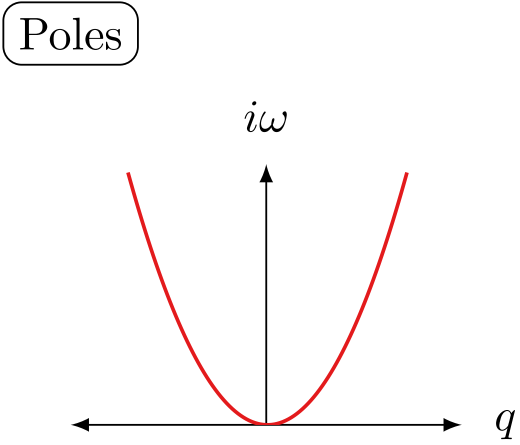
Hydrodynamics Example

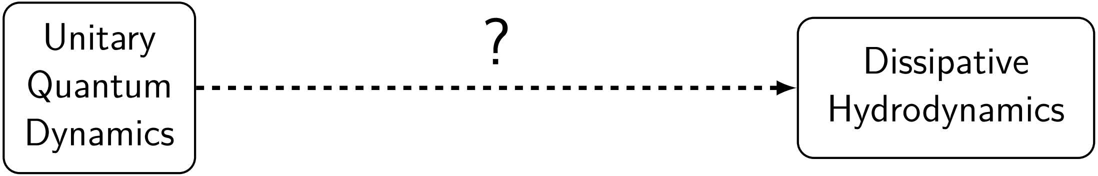
- Say $\epsilon(t,x)$ is the average energy density at a point x.
- Then energy diffusion is

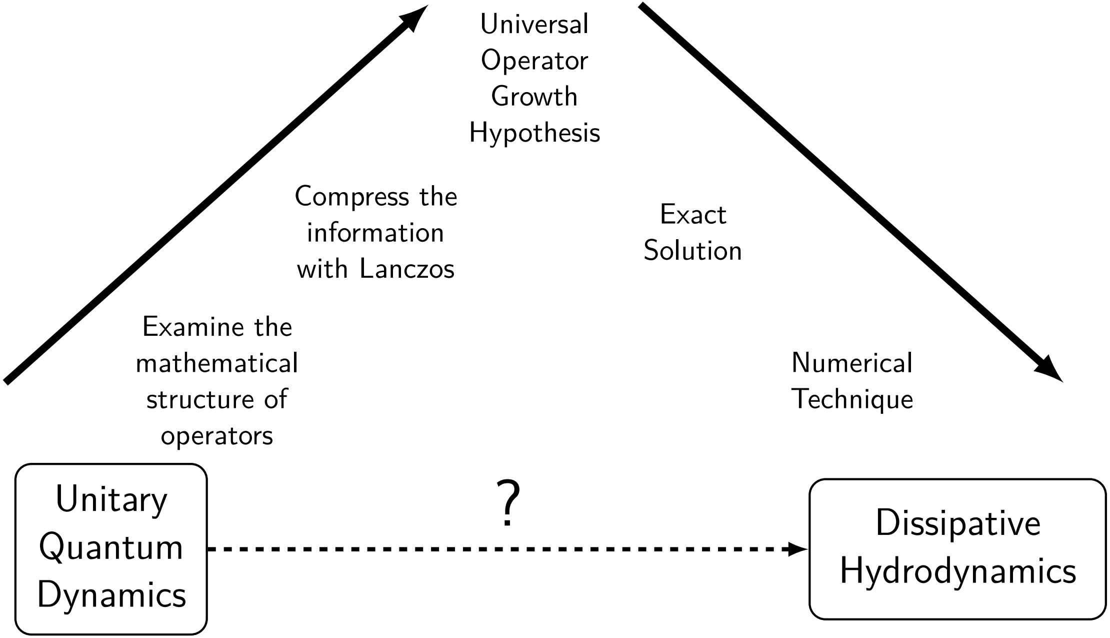
$$\frac{d}{dt}\epsilon(t,x) = D\nabla^2\epsilon(t,x) + \nabla f$$

  for D diffusion constant, f thermal noise
- Solved by the Green's function

$$G(\omega,q) = \frac{1}{i\omega + Dq^2}$$




Unitary Quantum Dynamics
?
Dissipative Hydrodynamics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The Graph of Operators

Let's consider an example. Suppose $\mathcal{O} = X_1$,

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i.$$

We know

$$\mathcal{O}(t) = e^{-iHt} \mathcal{O} e^{iHt}$$
$$= \mathcal{O} - it[H, \mathcal{O}] + (-it)^2 [H, [H, \mathcal{O}]] + \cdots$$

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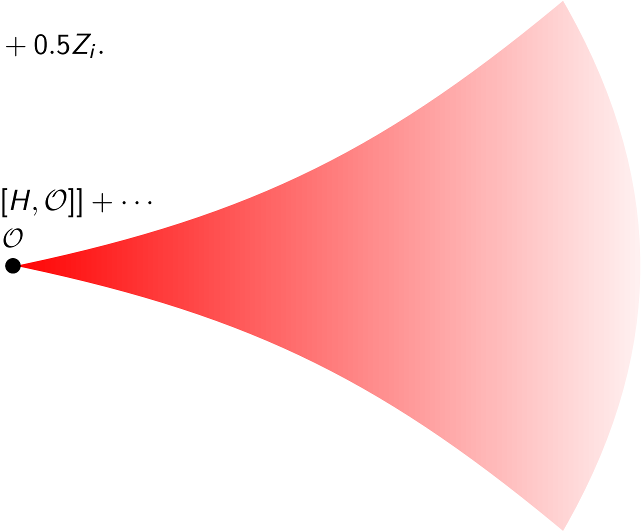

Let's compute!

$$\mathcal{O} = X_1$$

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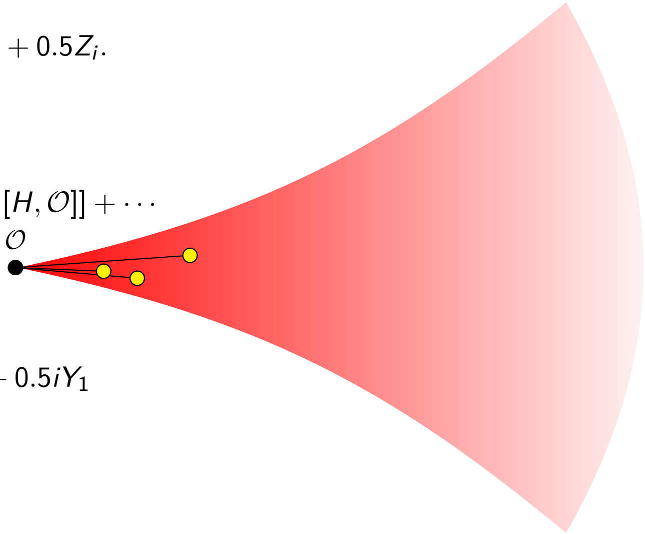

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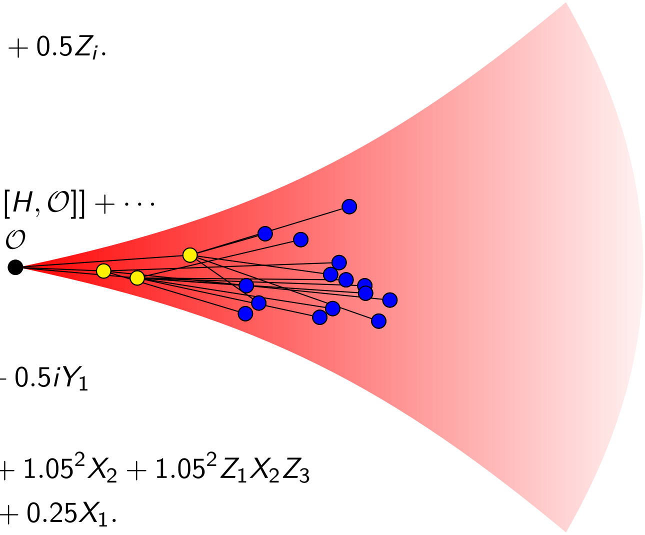

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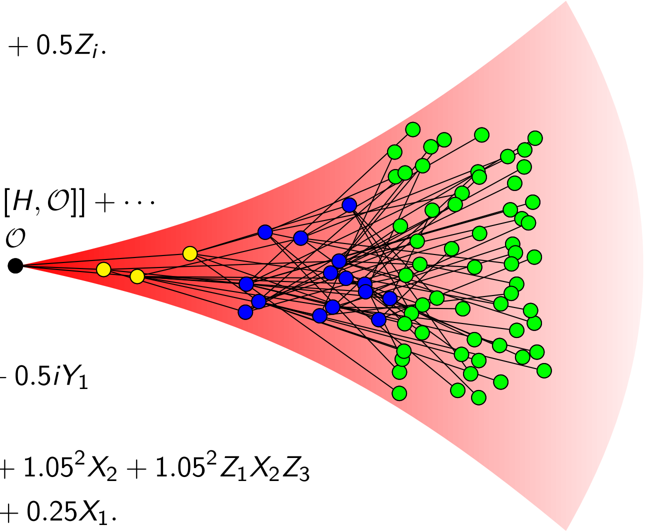

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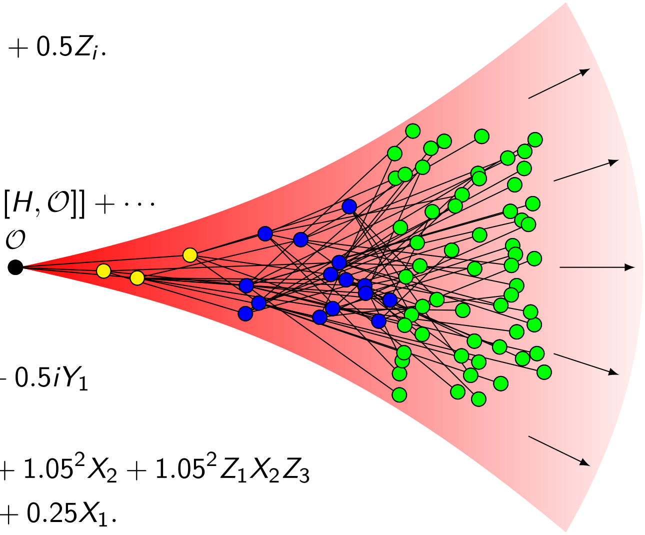

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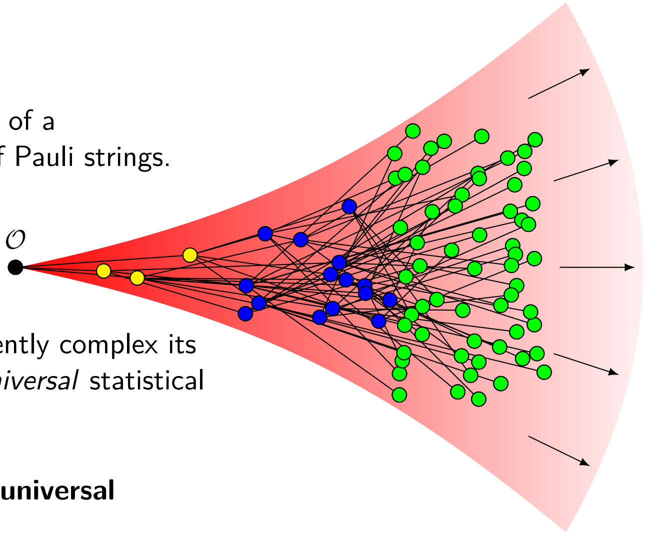
&\quad + 0.525X_1Z_2 + 0.525Z_1X_2 + 0.25X_1.
\end{aligned}$$

The Basic Idea

Operators flow from simple to complex, eventually becoming too complex to compute.

Complex operators are superpositions of a **thermodynamically large** number of Pauli strings.



So when an operator becomes sufficiently complex its dynamics should be governed by a *universal* statistical description.

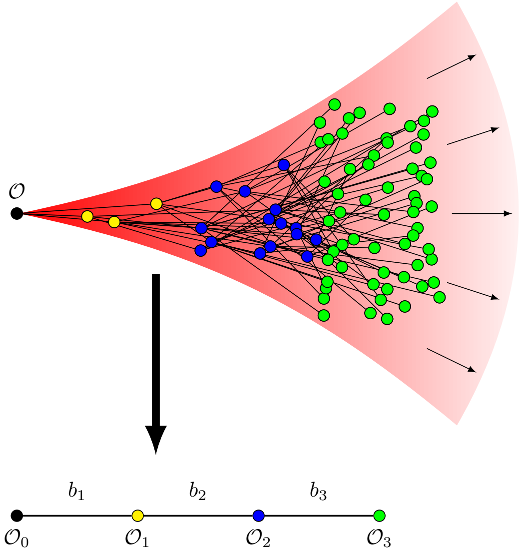
Our goal now is to formulate this universal description.

Simplifying the Graph

We have a hard problem here: quantum mechanics on an infinite graph.

Let's solve an easy problem instead: quantum mechanics on a 1d chain.

To tame the huge space of operators, we compress the information in it via the **Lanczos Algorithm**.

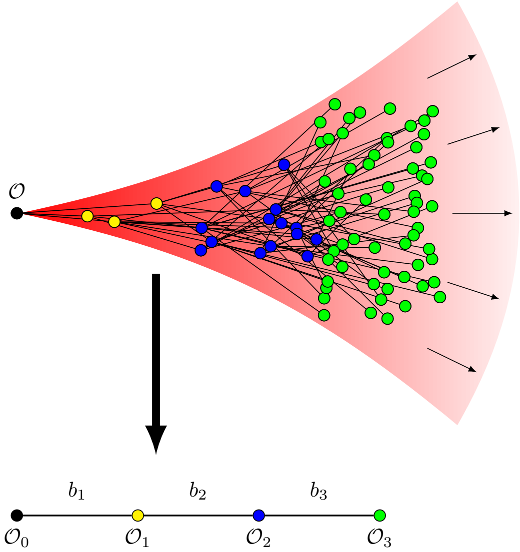


Krylov Vectors

- The Liouvillian is $\mathcal{L} := [H, \cdot]$.
- Take the sequence $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \dots\}$ and apply Gram-Schmidt to produce an orthogonal basis $\{\mathcal{O}_0 = \mathcal{O}, \mathcal{O}_1, \mathcal{O}_2, \dots\}$.
- The Liouvillian is tridiagonal in this basis

$$L_{nm} := \mathrm{Tr}[\mathcal{O}_n^\dagger \mathcal{L} O_m] = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

- The b_n's are called **Lanczos coefficients** and the $\mathcal{O}_n$'s are called **Krylov vectors**.



Viswanath & Müller, *The Recursion Method*, 2008.

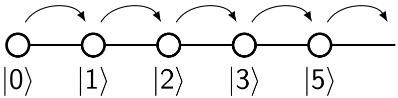
Familiar Tridiagonal Matrices

Tridiagonal matrices describe (single-body) 1D quantum mechanics problems.

The Raising Operator

$a^\dagger$ as in $H = a^\dagger a + \frac{1}{2}$

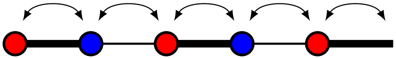
$$a^\dagger = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$



Rice-Mele Model

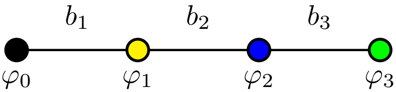
$$H = \sum_n \left[t + (-1)^n \delta\right] c_{n+1}^\dagger c_n + c^\dagger c_{n+1}$$
$$+(-1)^n \Delta c_n^\dagger c_n$$

$$H = \begin{pmatrix} \Delta & t+\delta & 0 & 0 & \cdots \\ t+\delta & -\Delta & t-\delta & 0 & \cdots \\ 0 & t-\delta & \Delta & t+\delta & \cdots \\ 0 & 0 & t+\delta & -\Delta & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$



The 1D Quantum Mechanics Problem

Define the **1D wavefunction** by $\varphi_n(t) := \mathrm{Tr}[\mathcal{O}(t)\mathcal{O}_n]$.



The operator evolves as $-i\frac{d}{dt}\mathcal{O} = \mathcal{L}\mathcal{O}$, and $\mathcal{L}$ is tridiagonal:

$$i\partial_t\varphi_n = -b_{n+1}\varphi_{n+} + b_n\varphi_{n-1}, \quad \varphi_n(0) = \delta_{n0}.$$

The autocorrelation function is probability of returning to the zeroth site at time t

$$C(t) = \mathrm{Tr}[\mathcal{O}(t)\mathcal{O}(0)] = \varphi_0(t).$$

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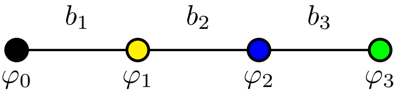

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Hamiltonian & Operator

Operator Graph

b_n's, $n = 1, 2, \ldots, \infty$

1D Quantum Mechanics

Hydrodynamics

Viswanath & Müller, *The Recursion Method*, 2008.

Examples

Let's try this for a variety of Hamiltonians.

$$H_1 = \sum_i X_i X_{i+1} + 0.709 Z_i + 0.9045 X_i$$

$$H_2 = H_1 + \sum_i 0.2 Y_i$$

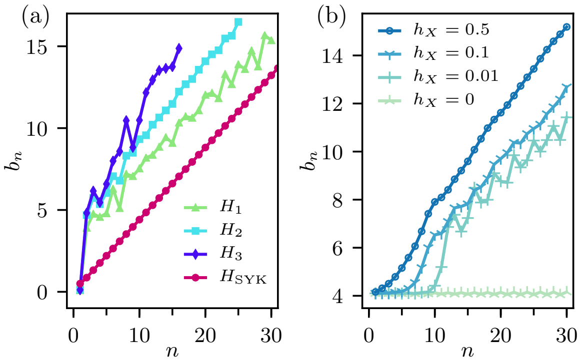
$$H_3 = H_1 + \sum_i 0.2 Z_i Z_{i+1}$$

$$H(h_X) = \sum_i X_i X_{i+1} - 1.05 Z_i + h_x X_i$$

$$H_{\text{SYK}}^{(q)} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \gamma_{i_1} \cdots \gamma_{i_q},$$

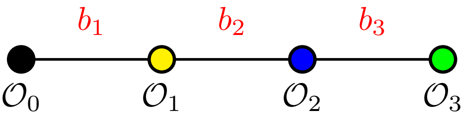
$$\overline{J^2_{i_1 \ldots i_q}} = 0,$$

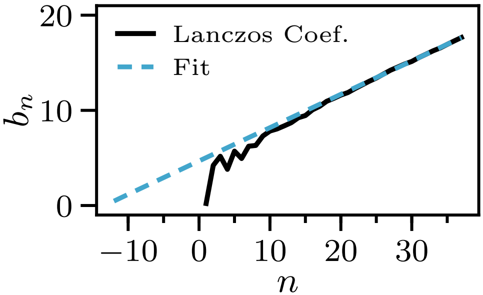
$$\overline{J^2_{i_1 \ldots i_q}}^2 = \frac{(q-1)!}{N^{q-1}} J^2$$



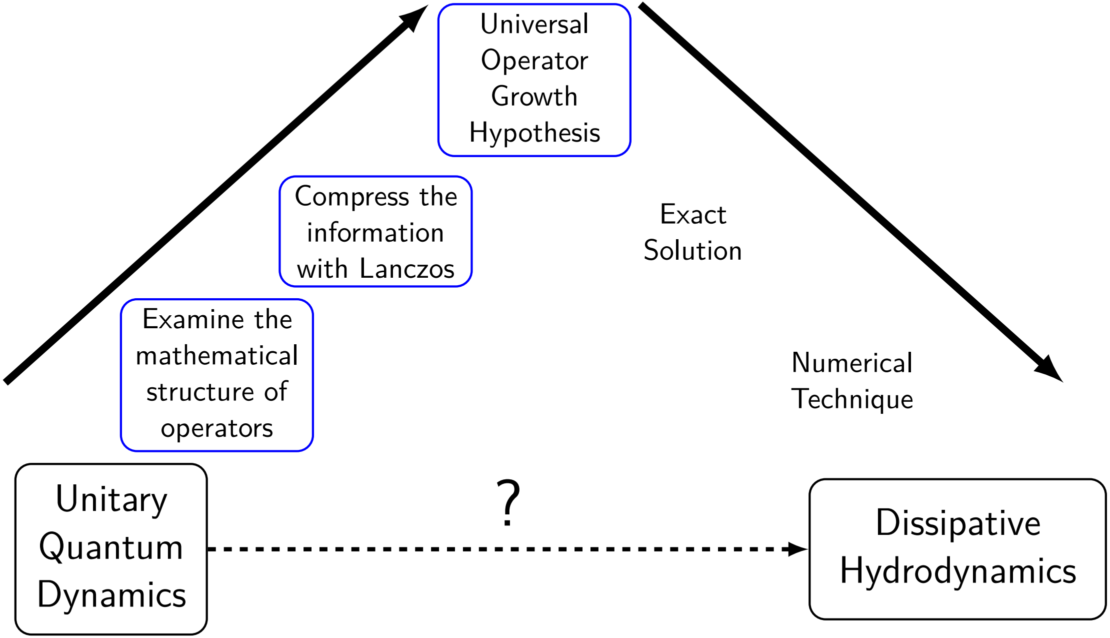
Hypothesis: In a chaotic[1] quantum system, the Lanczos coefficients b_n are asymptotically linear, i.e. for $\alpha, \gamma \geq 0$,

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$





[1] i.e. not quantum-integrable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An Exact Solution

We have a hypothesis, but what does it mean? To find out, let's study an exact solution to the hypothesis

- Consider

$$\widetilde{b}_n := \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \gamma.$$

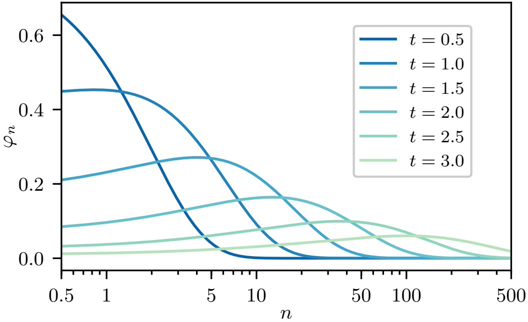
- We can solve this exactly:

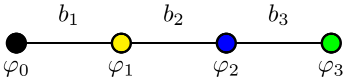
$$\varphi_n(t) = \sqrt{\frac{(\eta)_n}{n!}}\tanh(\alpha t)^n \operatorname{sech}(\alpha t)^\eta$$

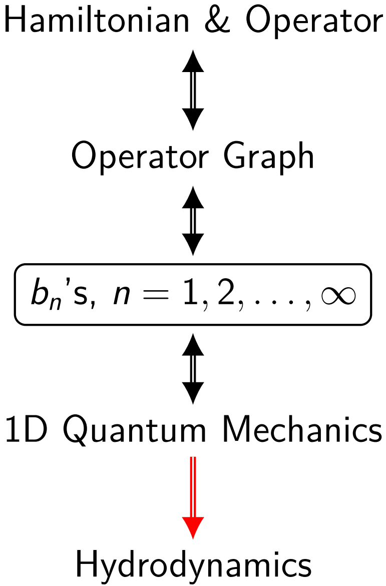
where $(\eta)_n = \eta(\eta+1)\cdots(\eta+n+1)$.

- Expected "position" in the 1D chain is

$$\langle n(t)\rangle = \eta\sinh(\alpha t)^2 \sim e^{2\alpha t}.$$



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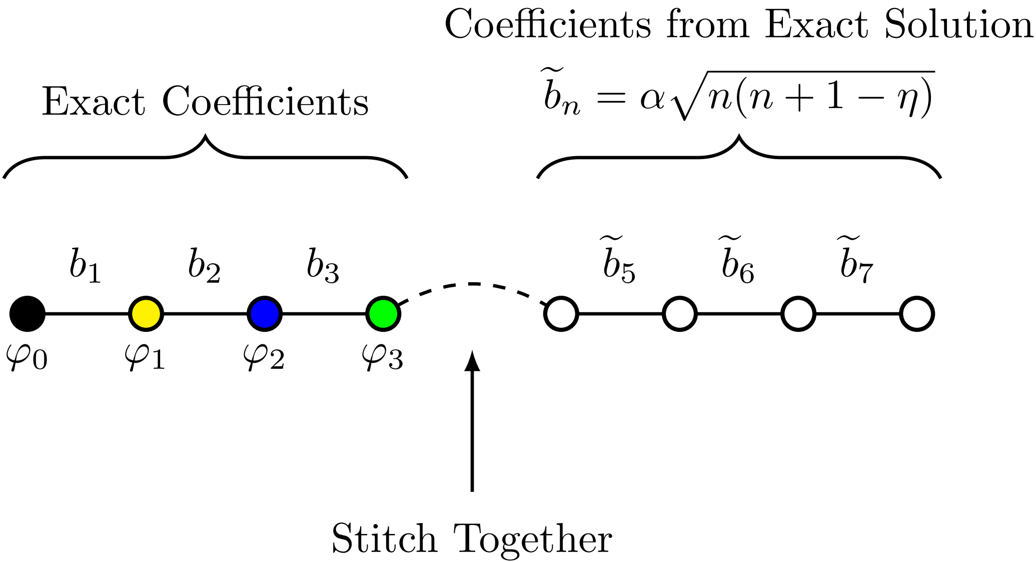
Numerical Method

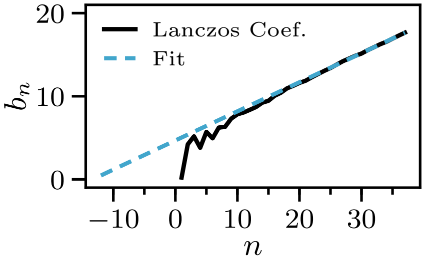


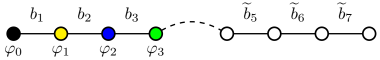
Algorithm

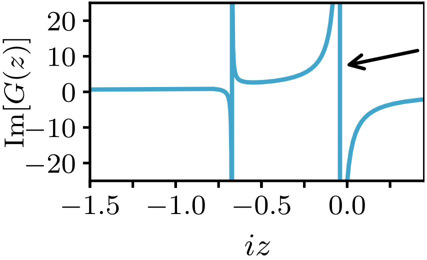
1. Compute $b_1, b_2, \ldots, b_N$ exactly and fit α and η to the exact solution

2. Stitch together the b_n's and the exact solution to find the Greens's function via the continued fraction expansion of the Green's function.

3. Identify the pole closest to the origin to extract diffusion.







Diffusion in the Chaotic Ising Model

- Chaotic Ising Model

$$H = \sum_j X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j$$

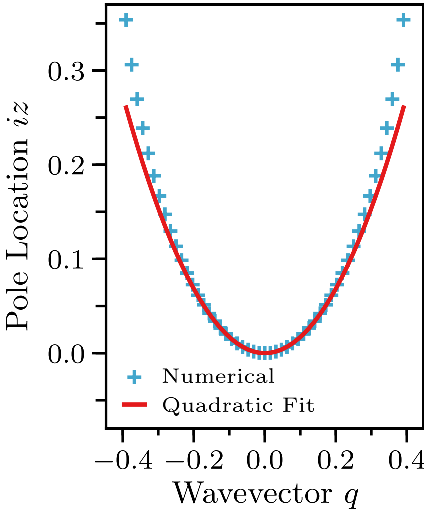
- Use initial operators at a range of wavevectors q

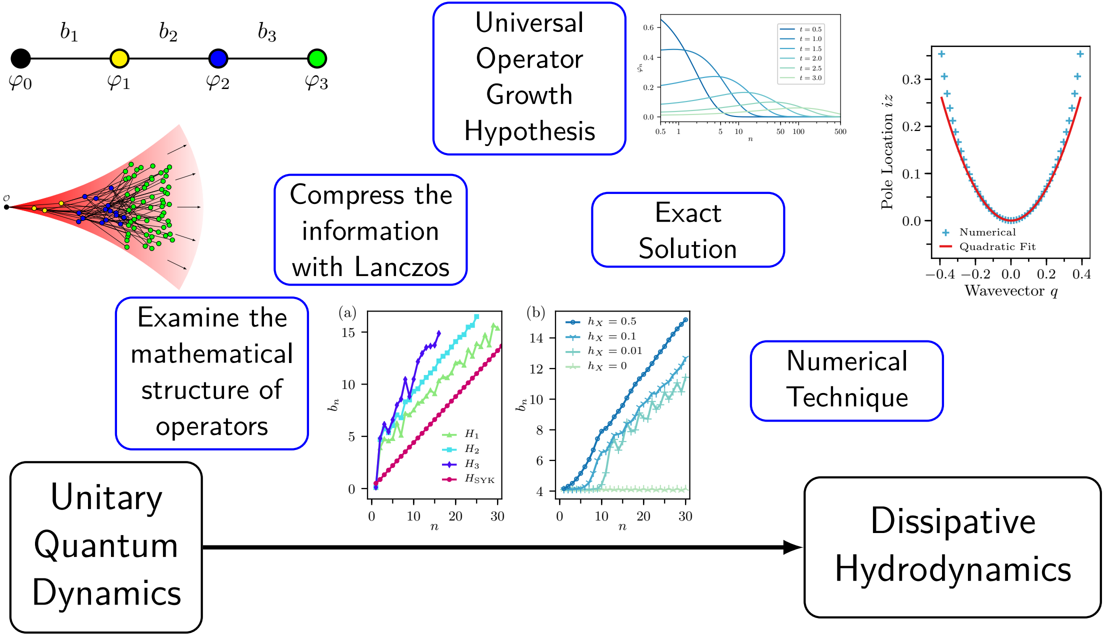
$$\mathcal{O}_q = \sum_j e^{iqj} \left(X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j \right)$$

- We see the dispersion relation for the diffusion equation.

$$\frac{d}{dt}\epsilon(t,x) = D\nabla^2\epsilon(t,x) + \nabla f.$$

- Fitting shows that $D = 3.35$.





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$t = 3.0$
φ_n
n
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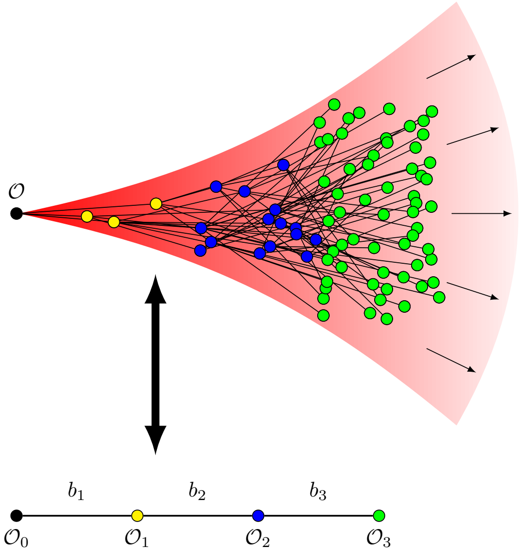
Wavevector q
Numerical
Quadratic Fit
(a)
(b)
b_n
H_1
H_2
H_3
H_{SYK}
$h_X = 0.5$
$h_X = 0.1$
$h_X = 0.01$
$h_X = 0$

Increasing Complexity

- When the hypothesis holds, then wavefunction spreads out exponentially in the 1D chain.

$$(n(t)) \sim e^{2\alpha t}$$

- Back in the graph, this means that the wavefunction is "escaping" towards more and more complicated operators.
- Therefore operators inevitably "escape" to higher complexity over time with rate 2α.
- Complex operators, far out in the graph, serve as a *thermodynamic bath* for simple operators, giving effective irreversible dynamics and quantum chaos.



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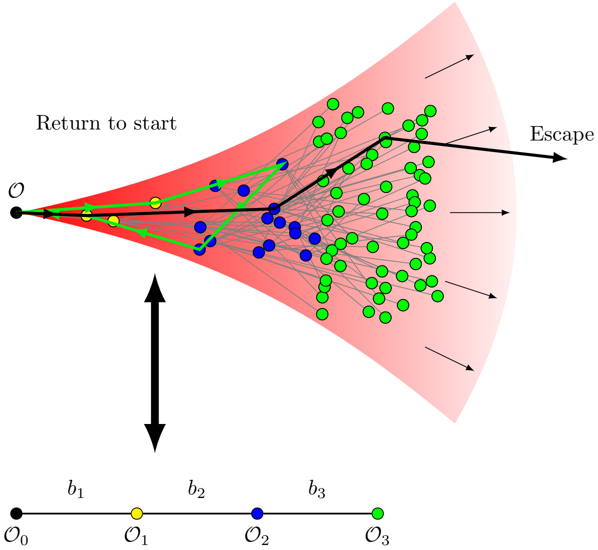

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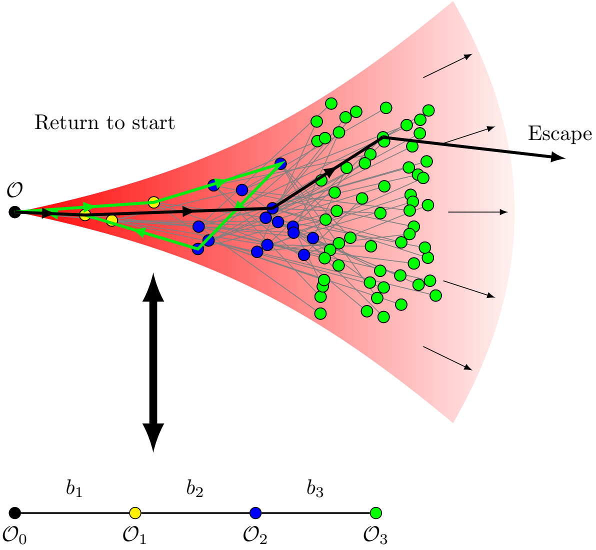
- Therefore operators inevitably “escape” to higher complexity over time with rate 2α.
- Complex operators, far out in the graph, serve as a *thermodynamic bath* for simple operators, giving effective irreversible dynamics and quantum chaos.



SYK-q	2	3	4	7	10	∞
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/(2\mathcal{J})$	0	0.454	0.620	0.799	0.863	1

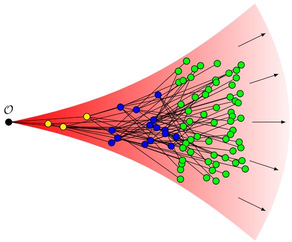
Roberts, Stanford, Streicher, 2018.

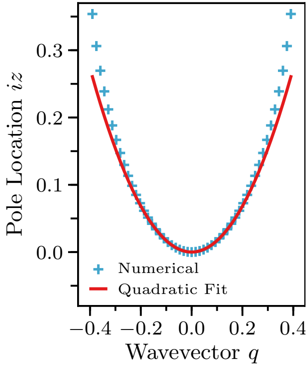
Summary

- The hypothesis governs operator growth in chaotic, closed quantum systems

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$

- Complexity growth leads to effectively irreversible dynamics.
- This gives a new numerical technique for computing hydrodynamics.
- The **operator growth rate** α controls the growth of complexity and chaos in quantum systems.
- arXiv:1812.08657





Extra Slide: Other Results

- Other guises of α: relation to the spectral function, analytic structure of $C(t)$, experimental probes.
- The exponential growth of Krylov complexity suggests that 2α can be interpretted as a Lyapunov exponent.
- Formal of complexity: Krylov-complexity and Q-complexities
- Theorem: Krylov complexity grows faster than any other complexity, including operator size and OTOCs.
- The theorem above implies the so-called "quantum bound on chaos" at low temperatures.
- Most of this story carries over directly to the classical case.
- One can show the SYK model obeys the hypothesis directly, and compute most of these quantities exactly.

Extra Slide: Operator Space

We move up a level of abstraction from the space of states to the **space of operators**.

- Operators $\mathcal{O}$ are now "kets", $|\mathcal{O})$.
- e.g. $|\mathcal{O}) = X_1 \otimes Y_2 \otimes Z_3 + 0.3 Y_1 \otimes X_2$.
- A basis of operators is the set of **Pauli Strings**

$$|\boldsymbol{\alpha}) = \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \otimes \sigma^{\alpha_n}$$

  for $\alpha_i = 0, 1, 2, 3$.
- Operator inner product:

$$(A|B) := \mathrm{Tr}[A^\dagger B].$$

- The **Liouvillian superoperator** gives the commutator of an operator against the Hamiltonian

$$\mathcal{L} = [H, \cdot].$$

- Heisenburg equation of motion

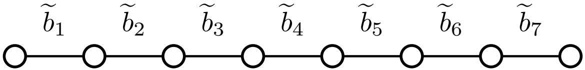
$$-i\frac{d\mathcal{O}}{dt} = [H, \mathcal{O}] \to -i\frac{d\,|\mathcal{O})}{dt} = \mathcal{L}\,|\mathcal{O})\,.$$

- By Baker-Campbell-Hausdorff,

$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} \to |\mathcal{O}(t)) = e^{-i\mathcal{L}t}\,|\mathcal{O})\,.$$

- Operators evolve in operator space like states in state space.

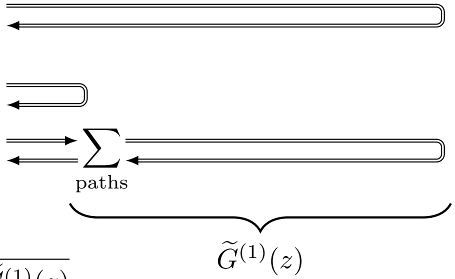
Extra Slide: The Recursion Method



$$\widetilde{G}(z) = \sum_{\text{paths}}$$

$$=$$

$$+$$



$$= \frac{1}{1 + \widetilde{b}_1^2 \widetilde{G}^{(1)}(z)}$$

$$= \cfrac{1}{1 + \cfrac{\widetilde{b}_1^2}{1 + \cfrac{\widetilde{b}_2^2}{1 + \widetilde{b}_3^2 \widetilde{G}^{(3)}(z)}}}$$