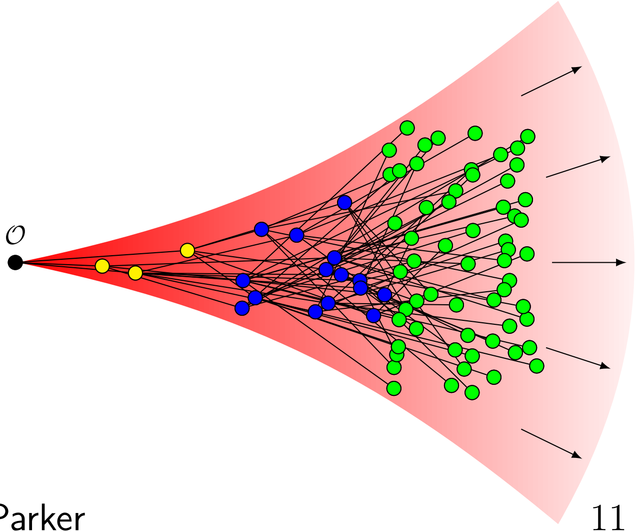
A Universal Operator Growth Hypothesis

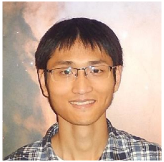


Daniel E. Parker

11 December 2019

Acknowledgements

Collaborators



Xiangyu Cao

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Advisor

Joel Moore

Quantum Mechanics

Microscopic description of the system.
Example: Chaotic Ising Model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i$$

Correlation functions:

$$C(t) = \langle \mathcal{O}(t,x)\mathcal{O}(0) \rangle$$

Hard Solution: Hamiltonian dynamics

$$\mathcal{O}(t) = e^{-iHt}\mathcal{O}e^{iHt}.$$

Exact and reversible dynamics.

Hydrodynamics

Macroscopic description of quantum systems as classical PDEs.

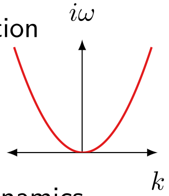
Example: Diffusion of energy

$$\frac{\partial}{\partial t}\varepsilon(t,x) = D\nabla^2\varepsilon(t,x) + \nabla f,$$

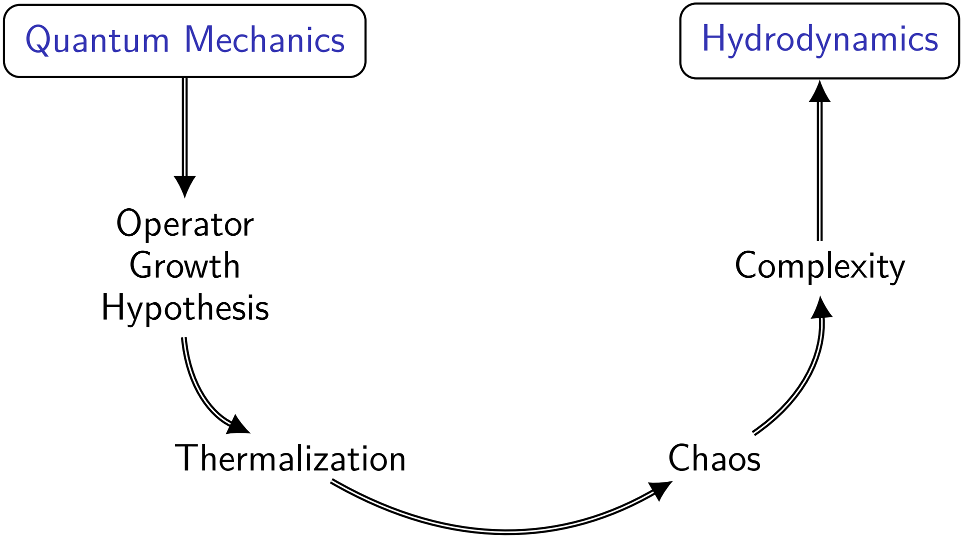
with D diffusion, f thermal noise.

Easy Solution: Green's function

$$G(i\omega, k) = \frac{1}{i\omega + Dq^2}$$



Approximate & irreversible dynamics.


Quantum Mechanics
Hydrodynamics
Operator Growth Hypothesis
Complexity
Thermalization
Chaos

Operator Growth

or

How I learned to stop worrying and love tridiagonalization.

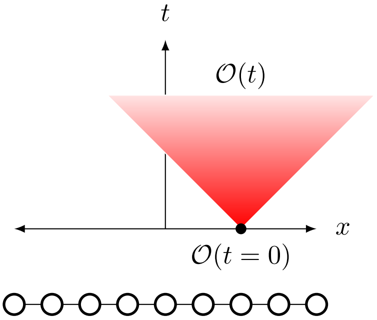
Operator Space

Inspiration: random unitary circuits.

- Keyserlingk, Rakovszky, Pollmann, Sondhi, 2017; Nahum, Vijay, Haah, 2017.
- Khemani, Vishwanath, Huse, 2017.

Consider a spin-1/2 system in d-dimensions with translation invariance.

$$H = \sum_{x \in \mathbb{Z}^d} h_x .$$



We abstract to the **space of operators**.

operators are "rounded" kets $|\mathcal{O})$

an example is $|O) = X_1 \otimes Y_2 \otimes Z_3 + 0.3 Y_1 \otimes X_2$

the inner product is $(A|B) := \mathrm{Tr}[A^\dagger B] / \mathrm{Tr}[1]$

the Liouvillian generalizes the Hamiltonian $\mathcal{L} = [H, \cdot]$.

time-evolution from Heisenberg EOM $-i \frac{d|\mathcal{O})}{dt} = \mathcal{L}\,|\mathcal{O})$.

solution $|\mathcal{O}(t)) = e^{i\mathcal{L}t}\,|\mathcal{O})$

Three Observables

A. Correlation Function

$$C(t) := (\mathcal{O}(t)|\mathcal{O}(0)) = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{(2n)!}(it)^{2n} \quad \text{with } \textbf{moments} \;\; \mu_{2n} = (\mathcal{O}|\mathcal{L}^{2n}|\mathcal{O})\,.$$

B. Green's Function

$$G(z) := (\mathcal{O}|\frac{1}{z-\mathcal{L}}|\mathcal{O}) = i\int_0^{\infty} e^{-izt}\,C(t)\,dt = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{z^{2n+1}}$$

C. Spectral Function

$$\Phi(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\,C(t)\,dt = \sum_{E,E'} \left|\langle E|\mathcal{O}|E'\rangle\right|^2 \delta(\omega-(E-E')).$$

The Graph of Operators

Example: Chaotic Ising Model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i.$$

Problem: Compute $C(t) = (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O})$.

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O} = \mathcal{O} + (it)\mathcal{L}\mathcal{O} + (it)^2\mathcal{L}^2\mathcal{O} + \cdots$$

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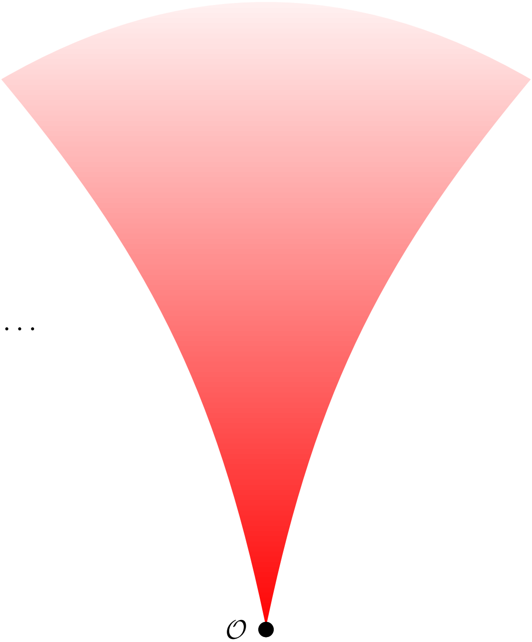
Let's compute!

$\mathcal{O} = X_1$



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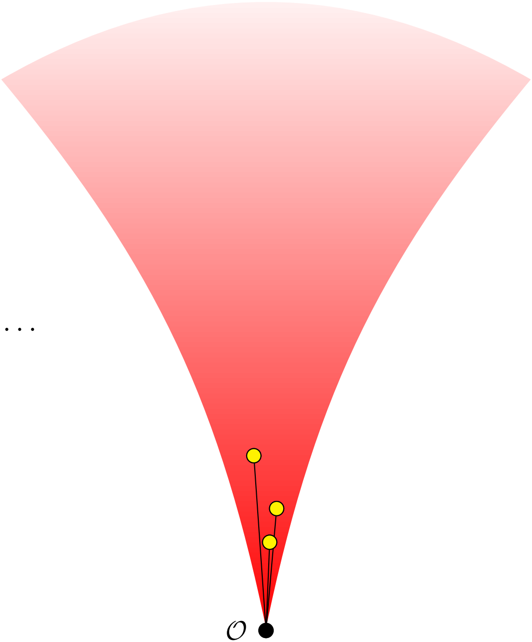
$$\mathcal{L}\mathcal{O} = 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1$$



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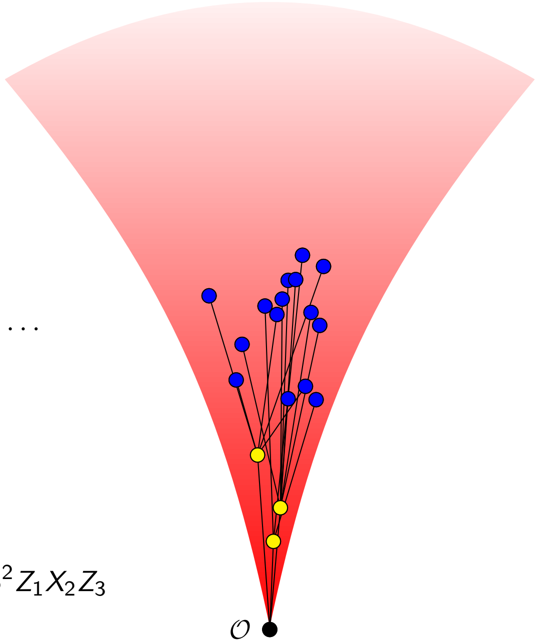
Let's compute!

$$\begin{aligned} \mathcal{O} &= X_1 \\ \mathcal{L}\mathcal{O} &= 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1 \\ \mathcal{L}^2\mathcal{O} &= 2.1 Z_1 Z_2 - 2.1 Y_1 Y_2 \\ &+ 1.05^2 Z_0 X_1 Z_2 + 1.05^2 X_1 + 1.05^2 X_2 + 1.05^2 Z_1 X_2 Z_3 \\ &+ 0.525 X_1 Z_2 + 0.525 Z_1 X_2 + 0.25 X_1. \end{aligned}$$



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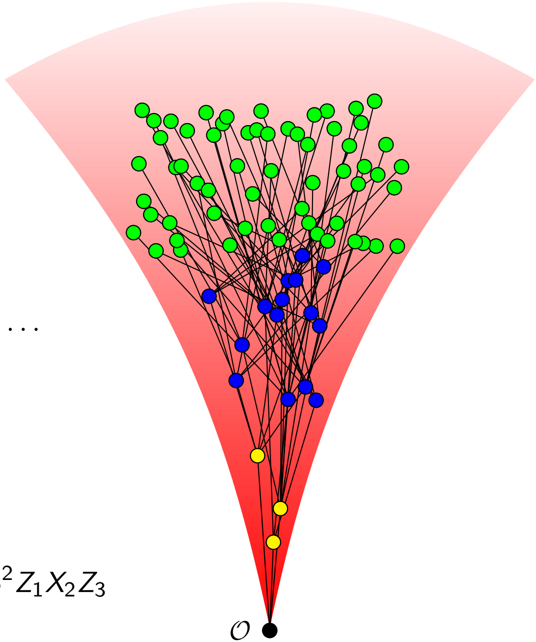

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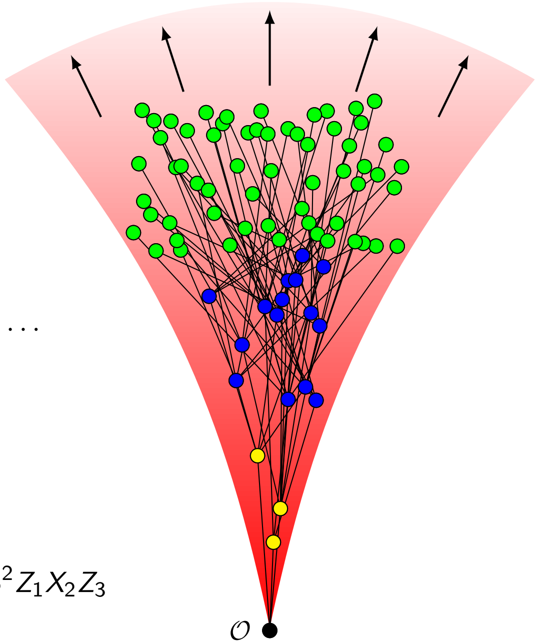
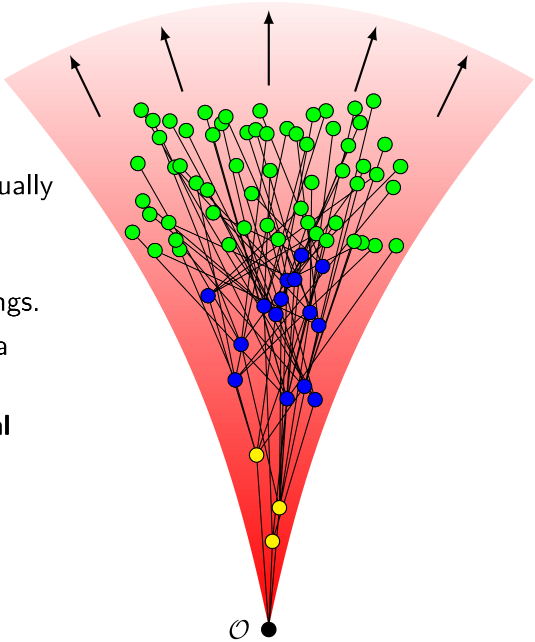


The Basic Idea

- Operators flow from simple to complex, eventually becoming too complex to compute.
- Complex operators are superpositions of a *thermodynamically large* number of Pauli strings.
- A sufficiently complex operator should admit a *universal* description.
- **Our goal now is to formulate this universal description.**



The Lanczos Algorithm

- Take the sequence $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \dots\}$ and apply Gram-Schmidt to orthogonalize $\{\mathcal{O}_0, \mathcal{O}_1, \mathcal{O}_2, \dots\}$.
- Explicitly, $|\mathcal{O}_1) := b_1^{-1}\mathcal{L}\,|O_0)$, $b_1 := (\mathcal{O}_0\mathcal{L}|\mathcal{L}\mathcal{O}_0)^{1/2}$,

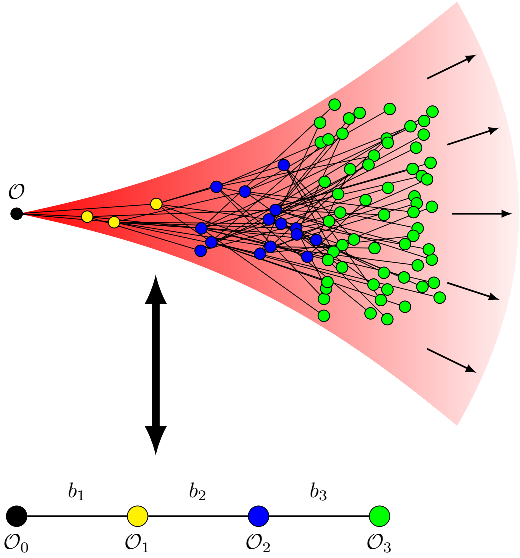
$$|A_n) := \mathcal{L}\,|\mathcal{O}_{n-1}) - b_{n-1}\,|\mathcal{O}_{n-2})\ ,$$

$$b_n := (A_n|A_n)^{1/2} \qquad \textbf{“Lanczos Coefficients”}$$

$$|\mathcal{O}_n) \ := \ b_n^{-1}\,|A_n) \qquad \textbf{“Krylov vectors”}$$

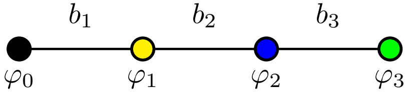
- The Liouvillian is tridiagonal in this basis

$$L_{nm} := (O_n^\dagger|\mathcal{L}|O_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$



D.C. Mattis, 1981; V.S. Viswanath & G. Müller, *The Recursion Method*, 2008.

The Recursion Method



Define the **1D wavefunction** by

$$|\mathcal{O}(t)) = \sum_{n=0}^{\infty} \varphi_n(t)\,|\mathcal{O}_n)\,, \quad \varphi_n(t) := (\mathcal{O}_n|\mathcal{O}(t))\,.$$

The operator evolves as $-i\frac{d}{dt}\mathcal{O} = \mathcal{L}\mathcal{O}$, and $\mathcal{L}$ is tridiagonal:

$$-i\partial_t \varphi_n = b_{n+1}\varphi_{n+1} + b_n \varphi_{n-1}, \quad \varphi_n(0) = \delta_{n0}.$$

The autocorrelation is just the wavefunction on site zero:

$$C(t) = (\mathcal{O}_0|\mathcal{O}(t)) = \varphi_0(t).$$

This is called the **recursion method** and dates back to the 1980s.

D.C. Mattis, 1981; V.S. Viswanath & G. Müller, *The Recursion Method*, 2008.

Encodings of Dynamics

A. Correlation Function

$$C(t) := (\mathcal{O}|e^{i\mathcal{L}t}|\mathcal{O}) = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{(2n)!)}(it)^{2n}$$

B. Green's Function

$$G(z) := (\mathcal{O}|\frac{1}{z-\mathcal{L}}|\mathcal{O}) = \sum_{n=0}^{\infty} \frac{\mu_{2n}}{z^{2n+1}}$$

C. Spectral Function

$$\Phi(\omega) := \sum_{E,E'} |\langle E|\mathcal{O}|E'\rangle|^2 \, \delta(\omega - (E - E'))$$

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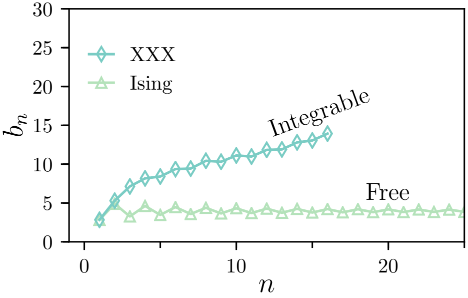
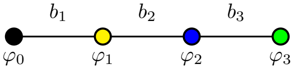
D. Lanczos Coefficients

$$\{b_n\}_{n=1}^{\infty} \ \& \ -i\partial_t \varphi_n = b_{n+1}\varphi_{n+1} + b_n \varphi_{n-1}$$

Empirical Patterns of Dynamics

Asymptotic	Growth Rate	System Type
$b_n \sim O(1)$	constant	Free models
$b_n \sim O(\sqrt{n})$	square-root	Integrable models
$b_n \sim$???	???	Chaotic models
$b_n \gtrsim O(n)$	superlinear	Disallowed





D.C. Mattis, 1981; V.S. Viswanath & G. Müller, *The Recursion Method*, 2008; Abanin, De Roeck, Ho, Huveneers, 2017.

Chaotic Examples

$$H_1 = \sum_i X_i X_{i+1} + 0.709 Z_i + 0.9045 X_i$$

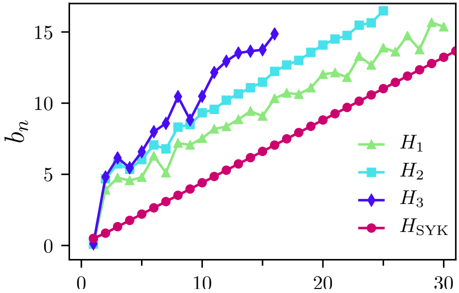
$$H_2 = H_1 + \sum_i 0.2 Y_i$$

$$H_3 = H_1 + \sum_i 0.2 Z_i Z_{i+1}$$

$$H_{\mathrm{SYK}}^{(q)} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \gamma_{i_1} \cdots \gamma_{i_q},$$

$$\overline{J^2_{i_1 \ldots i_q}} = 0,$$

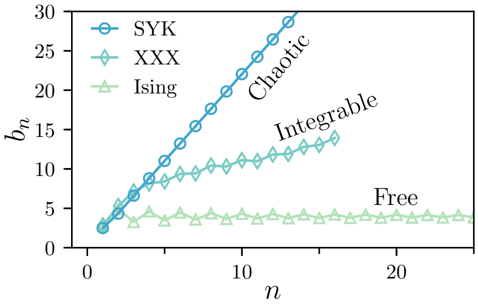
$$\overline{J^2_{i_1 \ldots i_q}}^2 = \frac{(q-1)!}{N^{q-1}} J^2$$



Hypothesis: In a chaotic[1] quantum system, the Lanczos coefficients b_n are asymptotically linear, i.e. for $\alpha, \gamma \geq 0$,

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$

Asymptotic	Growth Rate	System Type
$b_n \sim O(1)$	Constant	Free models
$b_n \sim O(\sqrt{n})$	Square-root	Integrable models
$b_n \sim O(n)$	Linear	Chaotic models
$b_n \gtrsim O(n)$	Superlinear	Disallowed



[1] i.e. non-integrable

Hypothesis: In a chaotic quantum system, the Lanczos coefficients have asymptotics

$$b_n = \begin{cases} A\frac{n}{W(n)} + O(1) \sim \frac{An}{\log n} & \text{if } d = 1 \\ \alpha n + \gamma + o(1) \sim \alpha n & \text{if } d \geq 2 \end{cases}$$

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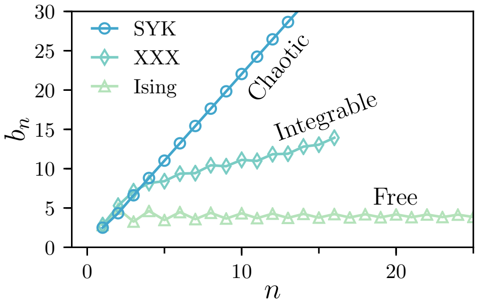


1D correction via Alex Avdoshkin & thesis of G.D. Bouch

Thermalization

The Hypothesis enforces "irreversible" dynamics.

Exact Asymptotic Behavior

Model

$$\widetilde{b}_n := \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \gamma.$$

Exact solution

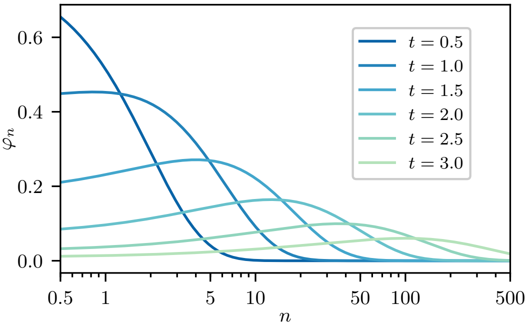
$$\widetilde{\varphi_n}(t) = \sqrt{\frac{(\eta)_n}{n!}}\tanh(\alpha t)^n \operatorname{sech}(\alpha t)^\eta$$

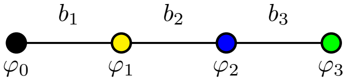
where $(\eta)_n = \eta(\eta+1)\cdots(\eta+n+1)$.

Define the **Krylov space position** operator

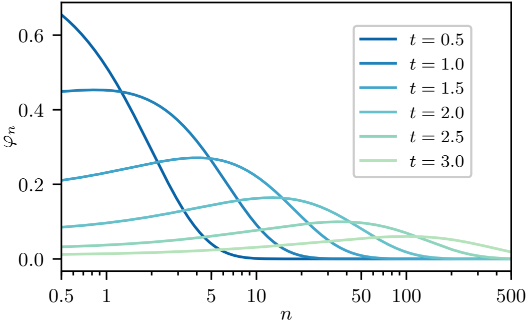
$$(n(t)) := \sum_{n\in\mathbb{N}} n\,|\varphi_n(t)|^2 = \eta\sinh(\alpha t)^2 \sim e^{2\alpha t}$$

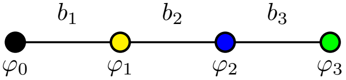
The wavefunction runs away “irreversibly” into the 1D chain.

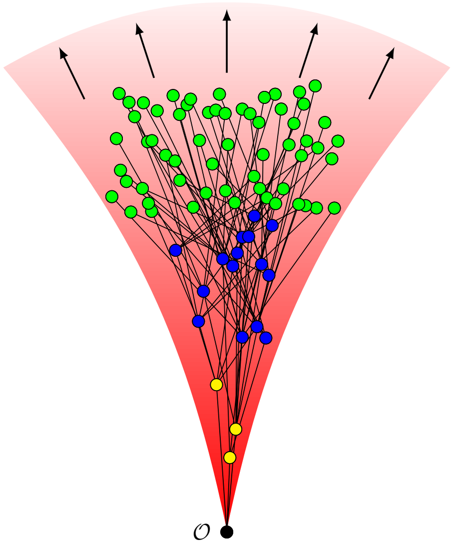




Picture of Thermalization

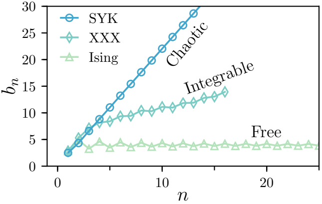


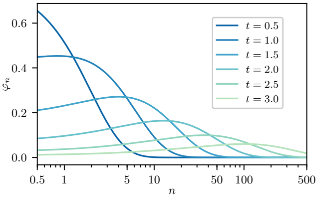




Four Observables

D. Lanczos Coefficients $b_n \sim \alpha n$





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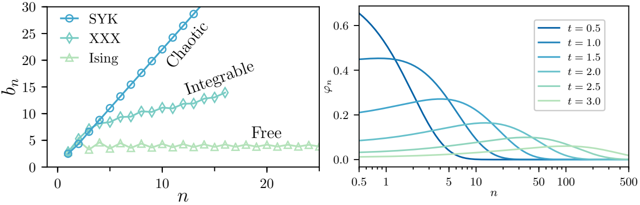
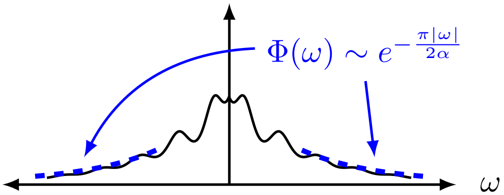


C. Spectral Function



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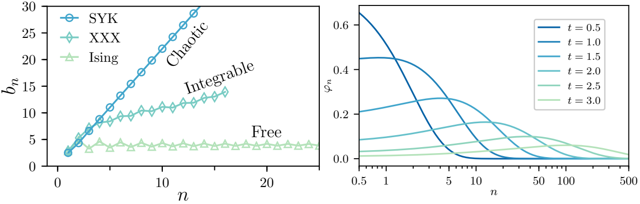

B. Green's Function

$$G(z) = \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{z - \cfrac{b_3^2}{\ddots}}}}$$

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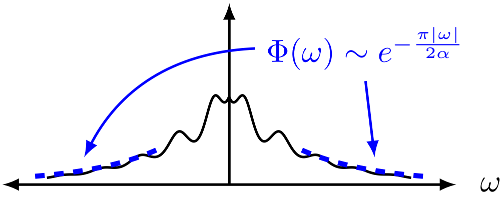

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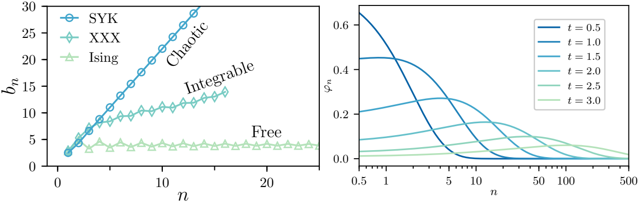

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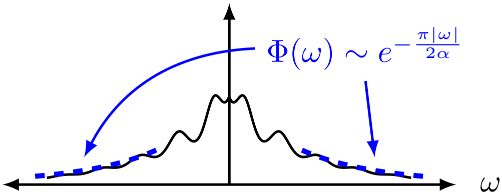
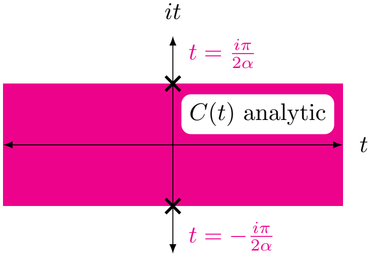


A. Correlation Function



Quantum Chaos

The Lanczos coefficients “diagnose” quantum chaos.

Hallmarks of Quantum Chaos

- **Level Statistics** give a "highly microscopic" indicator of chaos. With $\delta E_n := E_{n+1} - E_n$, the r-statistic is:

$$r = \overline{\frac{\min\{\delta E_n, \delta E_{n+1}\}}{\max\{\delta E_n, \delta E_{n+1}\}}} \approx \begin{cases} 0.386 & \text{integrable} \\ 0.530 & \text{chaotic} \end{cases}$$

- **Eigenstate Thermalization Hypothesis** predicts matrix elements of chaotic systems

$$\langle E_n|\mathcal{O}|E_m\rangle = \mathcal{O}(\overline{E})\delta_{nm} + e^{-S(\overline{E})} f_{\mathcal{O}}(\overline{E}, \omega) R_{nm}$$

Dynamics	Level Stats.	b_n	$\Phi(\omega)$ or $\lvert f_{\mathcal{O}}\rvert^2$
Free	-	$O(1)$	$\theta(\lvert\omega - 2W\rvert)$
Integrable	Poisson	$O(\sqrt{n})$	$O(e^{-\omega^2})$
Chaotic	GOE	$O(n)$	$O(e^{-\omega})$

- **Exponential sensitivity**

Example 1: Chaotic Ising Model

Model

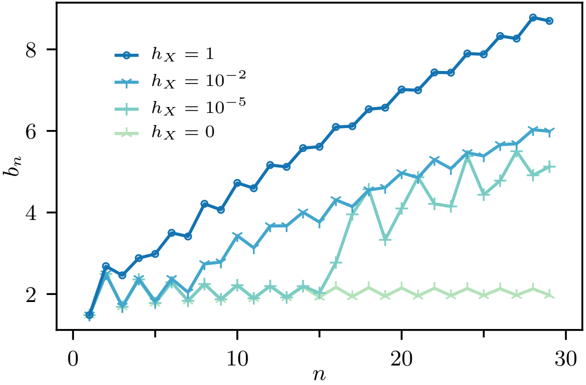
$$H = \sum_i J\left[X_i X_{i+1} + hZ_i\right] + h_x X_i$$

Dynamics In the thermodynamic limit,

$$\begin{cases} h_x = 0 & \text{free model} \\ h_x > 0 & \text{chaotic model.} \end{cases}$$

Perturbation Theory Resonances appear at order $O(h_x/J)$.

$$b_n \approx \begin{cases} O(1) & n < O(h_x/J) \\ O(n) & n > O(h_x/J) \end{cases}$$



Example 2: SYK Model

Model[1]

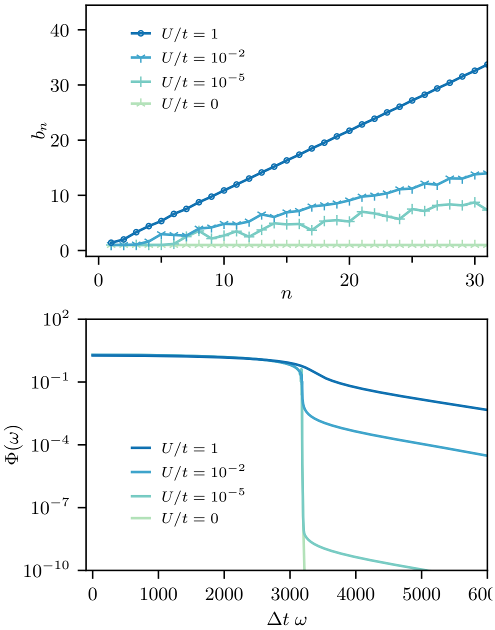
$$H = tH_{\mathrm{SYK}}^{(2)} + UH_{\mathrm{SYK}}^{(4)}$$

$$H_{\mathrm{SYK}}^{(q)} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 \ldots i_q} \gamma_{i_1} \cdots \gamma_{i_q},$$

$$\overline{J_{i_1 \ldots i_q}^2} = 0, \quad \overline{J_{i_1 \ldots i_q}^2}^2 = \frac{(q-1)!}{N^{q-1}} J^2$$

Dynamics

$$q = \begin{cases} 2 & \text{free model} \\ 4, 6, 8, \ldots & \text{chaotic model} \end{cases}$$



[1] Sachdev, Ye, 1993; Parcollet, Georges, 1999; Kitaev, 2015; Maldacena, Stanford, 2016, etc.

Example 3: XXZ+NNN

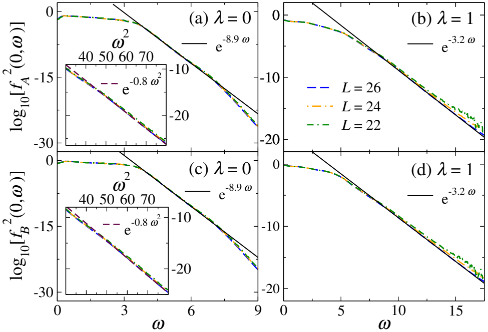
Model

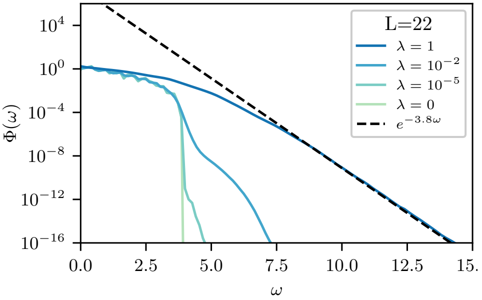
$$H_1 = \sum_i X_i X_{i+1} + Y_i Y_{i+1} + \Delta Z_i Z_{i+1}$$
$$+ \lambda \left(\sum_i X_i X_{i+2} + Y_i Y_{i+2} + \Delta Z_i Z_{i+2} \right)$$

$$A := \sum_i Z_i Z_{i+1}, B := \sum_i S_i^+ S_{i+2}^- + \text{h.c.}$$

Dynamics

$$\begin{cases} \Delta = 0 \;\&\; \lambda = 0 & \text{free model} \\ \Delta \neq 0 \;\&\; \lambda = 0 & \text{integrable model} \\ \Delta \neq 0 \;\&\; \lambda \neq 0 & \text{chaotic model} \end{cases}$$





LeBlond, Mallayya, Vidmar, Rigol, 2019. arXiv:1909.09654.

Model	**Op.**	**Dynamics**	**Lanczos**	**Evidence**	**Ref.**
Ising	$\widehat{Z}$	Free	$O(1)$	Analytic	Viswanath & Müller
XX	$\widehat{Z}$	Free	$O(1)$	Analytic	Viswanath & Müller
$\mathrm{SYK}^{(2)}$	γ	Free	$O(1)$	Analytic	Maldacena, Shenkar, Stanford, 2016
XX	$\widehat{X}$	Free*	$O(\sqrt{n})$	Analytic	Viswanath & Müller
Free Fermions in Disguise	$\widehat{Z}$	Free*	$O(\sqrt{n})$	Numerical	see Fendley, 2019.
MBL	$\widehat{Z}$	Int.	$O(\sqrt{n})$	Numerical	
XXZ	$\widehat{Z}$	Int.	$O(\sqrt{n})$	Numerical	
Chaotic Ising	$\widehat{Z}$	Chaotic	$O(n)$	Numerical	
XXZ + NNN	$\widehat{ZZ}$	Chaotic	$O(n)$	Numerical	LeBlond, Mallayya, Vidmar, Rigol, 2019.
$\mathrm{SYK}^{(4)}$	γ	Chaotic	$O(n)$	Numerical	
$\mathrm{SYK}^{(\infty)}$	γ	Chaotic	$O(n)$	Analytic	Roberts, Stanford, Streicher, 2018.
SYK Hopping	γ	Chaotic	$O(n)$	Analytic	
2D Fermi Hubbard	$\widehat{J}$	Chaotic	$O(n)$	Numerical	Huang, private comm.
Bouch Model	$\widehat{X}$	Chaotic	$O(n)$	Analytic	Bouch, 2015

Complexity

The Lanczos coefficients quantify quantum chaos.

K-Complexity

I will introduce a measure of quantum chaos ("**K-Complexity**") that is

1. easy to interpret
2. easy to compute
3. works in all quantum systems (not semiclassical).

Exponential Sensitivity

- A hallmark of chaos is *exponential sensitivity* to small perturbations.
- Classically, this is measured by the Lyapunov exponent.

[1] Kitaev, 2014. [2] Maldacena, Shenker, Stanford, 2016.

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- The **Out-of-time-order commutator** generalizes the Lyapunov exponent λ_L to semi-classical systems[1]

$$\mathrm{OTOC}(t) := ([\mathcal{O}(t), V][\mathcal{O}(t), V]) \sim e^{\lambda_L t}.$$

and λ_L is called the **quantum Lyapunov exponent**.

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  and λ_L is called the **quantum Lyapunov exponent**.
- High-energy theorists have shown a "universal bound on chaos": for $T \to 0$,[2]

$$\lambda_L \leq 2\pi T. \tag{1}$$

[1] Kitaev, 2014. [2] Maldacena, Shenker, Stanford, 2016.

Out-of-time-order Confusion

Semiclassical OTOCs usually saturate at[1]

$$t = \begin{cases} O(\log(1/\hbar)) & \text{semiclassics} \\ O(\log N) & \text{large-}N \\ O(1) & \text{no small parameter} \end{cases}$$

Regularization Dependent Choice of norm:

$$(A|B)_\beta := \mathrm{Tr}[\rho A^\dagger B] \qquad \text{“Physical”}$$

$$(A|B)_\beta^W := \mathrm{Tr}[\rho^{1/2} A^\dagger \rho^{1/2} B] \qquad \text{“Wightman”}$$

where $\rho = e^{-\beta H}/Z$.

[1] Maldacena, Shenker, Stanford, 2018. [2] Khemani, Huse, Nahum, 2018.

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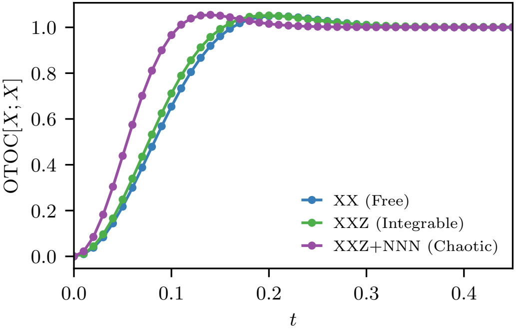

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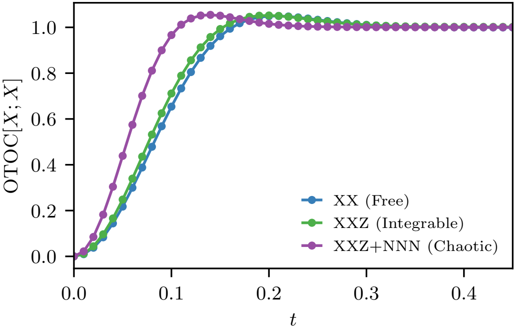

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where $\rho = e^{-\beta H}/Z$.

Therefore:

- Difficult to define λ_L in spin chains.
- Must use clever tricks like velocity-dependent Lyapunov exponents.[2]

[1] Maldacena, Shenker, Stanford, 2018. [2] Khemani, Huse, Nahum, 2018.

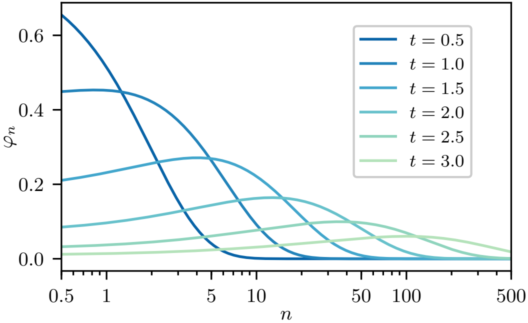
K-Complexity

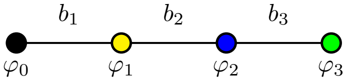
The Krylov vectors $\mathcal{O}_n$ grow successively larger, have more components, need more resources. . . they are more complex.

Therefore define the **K-Complexity** as

$$(n(t)) := \sum_{n\in\mathbb{N}} n\,|\varphi_n(t)|^2 \sim e^{2\alpha t}$$

where $\varphi_n(t) := (\mathcal{O}_n|\mathcal{O}(t))$.





[1] Roberts, Stanford, Streicher, 2018.

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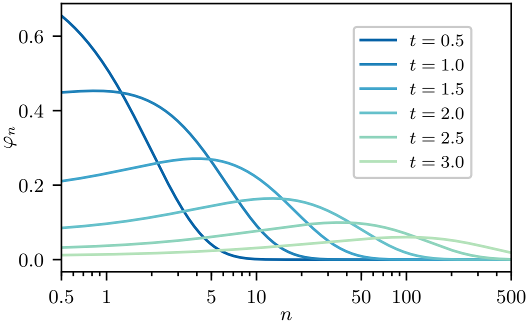
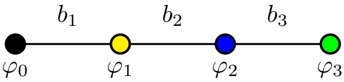


SYK-q	2	3	4	7	10	∞
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/(2\mathcal{J})$[1]	0	0.454	0.620	0.799	0.863	1



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Rigorous Bounds

Proposition: Suppose $T = \infty$. For any local operator, $\exists C > 0$ such that

$$\mathrm{OTOC}(t) \leq C \cdot (n(t)).$$

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$$\mathrm{OTOC}(t) \leq C \cdot (n(t)) \,.$$

Corollary: Suppose $b_n \asymp \alpha n$ and $T = \infty$. Then, if the quantum Lyapunov exponent λ_L is defined,

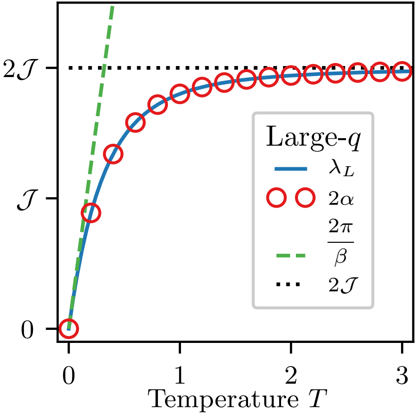
$$\lambda_L \leq 2\alpha.$$

Proposition: Suppose $b_n \asymp \alpha_W n$ using the Wightman regularization. Then

$$2\alpha \leq 2\pi T.$$

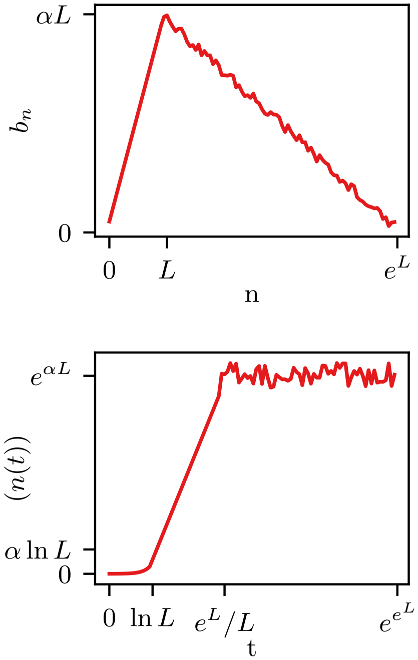
Conjecture: Under the same assumptions,

$$\lambda_L \leq 2\alpha \leq 2\pi T.$$



Finite Systems

- The Krylov operators $\mathcal{O}_n$ hit the edge at $n = O(L)$ and stop growing.
- Random matrix theory then kicks in and orthogonality causes the b_n's to decrease.
- The K-complexity keeps growing linearly until $t = O(e^L/L)$.
- This seems to match quantitatively with the "switchback effect" considered in black hole complexity.[1,2]



[1] Barbón, Rabinovici, Shir, Sinha, 2019. [2] DP *et al*, in progress.

Chaos and Complexity

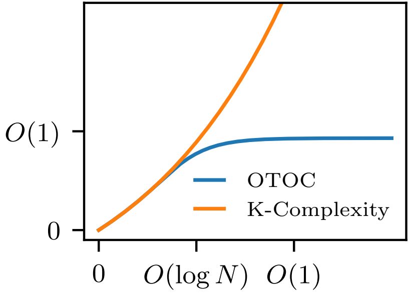
$$(n(t)) := \sum_{n\in\mathbb{N}} n\,|\varphi_n(t)|^2 \asymp e^{2\alpha t}$$

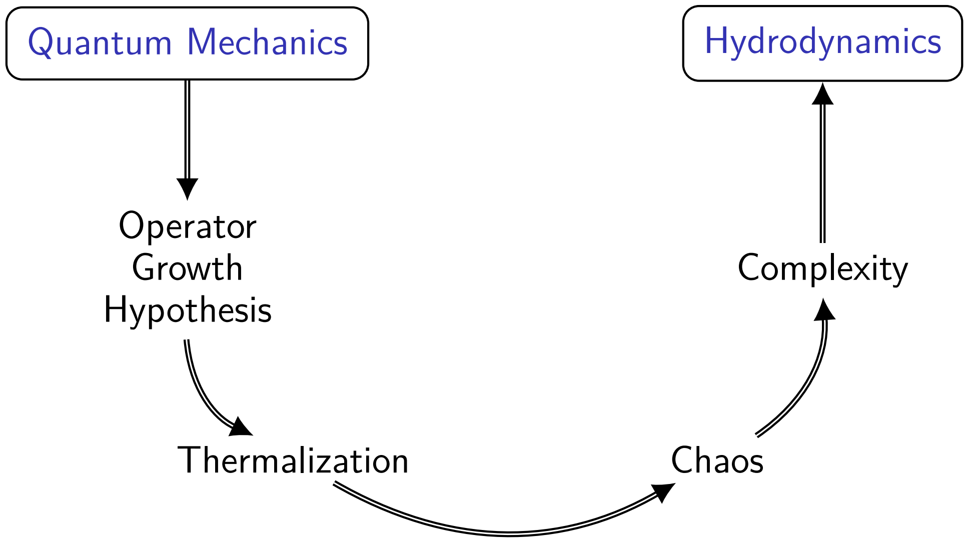
Dynamics	Level Stats.	b_n	$\Phi(\omega)$ or $\lvert f_{\mathcal{O}}\rvert^2$	K-complexity
Free	-	$O(1)$	$\theta(\lvert\omega - 2W\rvert)$	$(n)_t \asymp t$
Integrable	Poisson	$O(\sqrt{n})$	$O(e^{-\omega^2})$	$(n)_t \asymp t^2$
Chaotic	GOE	$O(n)$	$O(e^{-\pi\omega/2\alpha})$	$(n)_t \asymp e^{2\alpha t}$

Therefore α is the complexity growth rate.

This is experimentally observable from high-frequency heating.

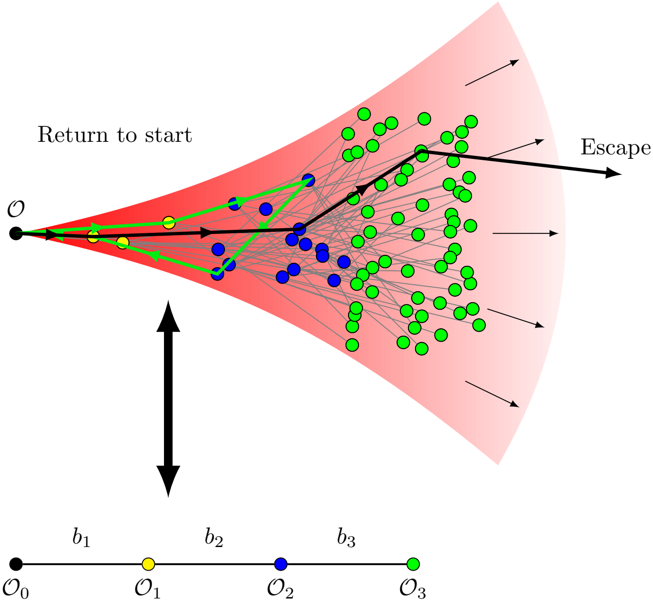
The complexity is non-perturbative data needed to compute hydrodynamics.



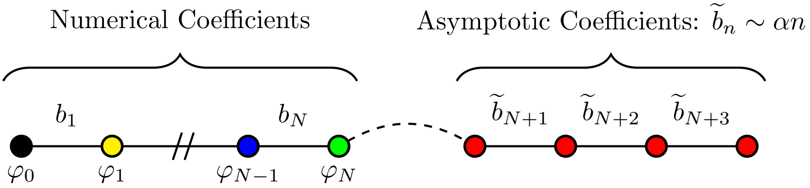

Quantum Mechanics
Hydrodynamics
Operator
Growth
Hypothesis
Complexity
Thermalization
Chaos

Hydrodynamics

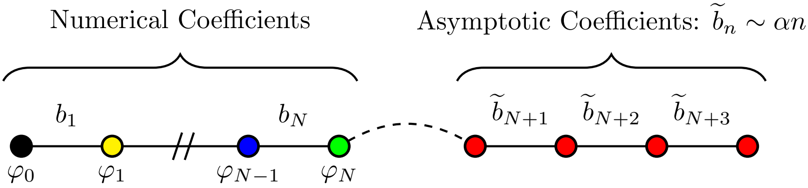
K-complexity gives rise to emergent hydrodynamics.



Return to start
$\mathcal{O}$
Escape
b_1
b_2
b_3
$\mathcal{O}_0$
$\mathcal{O}_1$
$\mathcal{O}_2$
$\mathcal{O}_3$



Numerical Coefficients
Asymptotic Coefficients: $\widetilde{b}_n \sim \alpha n$
b_1
b_N
$\widetilde{b}_{N+1}$
$\widetilde{b}_{N+2}$
$\widetilde{b}_{N+3}$
φ_0
φ_1
φ_{N-1}
φ_N



$$
L = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}
\approx \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & b_N \\ 0 & 0 & b_n & \widetilde{G^{(N)}}(z) \end{pmatrix}
$$

$$
G(z) = \int dt\, e^{izt} \langle \mathcal{O}(t)\mathcal{O}(0) \rangle
\approx \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{\cfrac{\ddots}{z - b_N^2 \widetilde{G^{(N)}}(z)}}}}
$$

$$
\widetilde{G^{(N)}}(z) = \Gamma(N+1)\Gamma\left(\frac{z+1}{2}\right) \times {}_2F_1\left(N+1, \frac{z+1}{2}, \frac{z+2N+3}{2}; -1\right)
$$

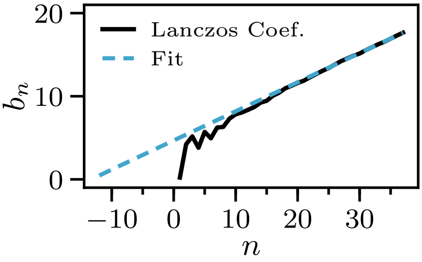
Algorithm

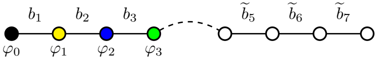
0. Choose a local operator $\mathcal{O}$ whose correlation $C(t) = \mathrm{Tr}[\mathcal{O}(t)\mathcal{O}(0)]$ should be hydrodynamical.

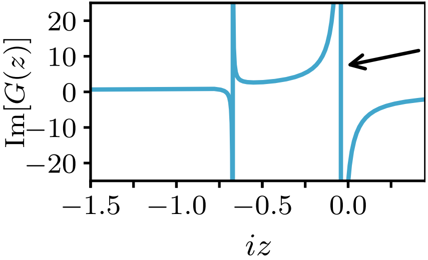
1. Compute $b_1, \ldots, b_N$ via infinite exact diagonalization and fit the slope α.

2. Stitch together the b_n's and the asymptotic solution $\widetilde{G^{(N)}}$.

3. Identify the pole closest to the origin to extract the hydrodynamical dispersion relation.







Diffusion in the Chaotic Ising Model

Chaotic Ising Model

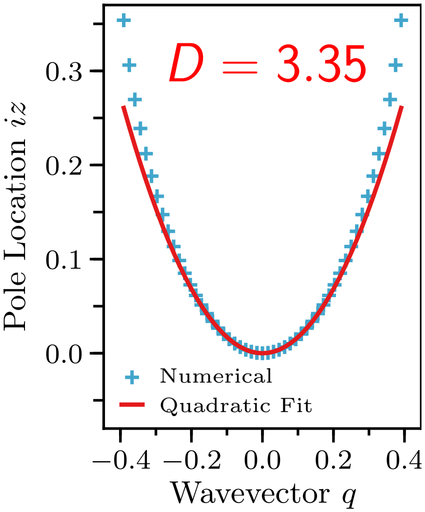
$$H = \sum_j X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j$$

Initial operator at wavevector k:

$$\mathcal{O}_k = \sum_j e^{ikj} \left(X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j \right)$$

We see the dispersion relation for diffusion

$$\frac{d}{dt}\epsilon(t,x) = D\nabla^2\epsilon(t,x).$$



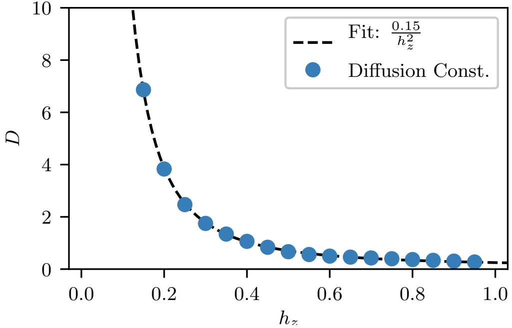
Diffusive Phenomenology

Chaotic Ising Model

$$H = \sum_j X_j + 1.05 Z_j Z_{j+1} + h_x Z_j$$

As $h_x \to 0$, a Drude peak ($D \to \infty$) emerges.

This is a practical method for computing hydrodynamics at strong coupling.



Finite Temperature

At $T < \infty$ there is a *physical* choice of innner product.
Suppose g is an even measure on the thermal circle:

1. $g : [0, \beta] \to \mathbb{R}$ (or a distribution)
2. $g(\lambda) = g(\beta - \lambda)$
3. $\int_0^\beta g(\lambda) = 1.$

Then, with $y := e^{-H}$, there is a g-inner product:

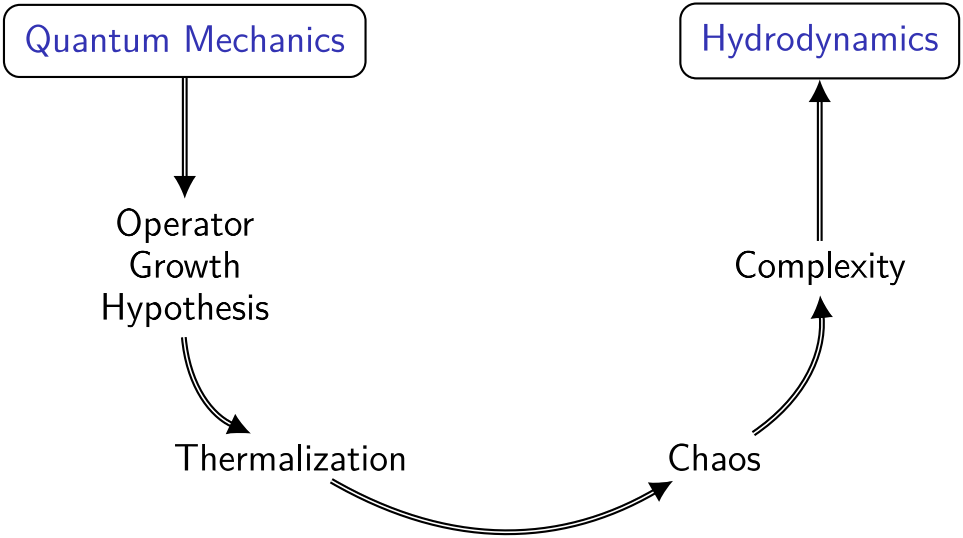
$$(A|B)_\beta^g := \frac{1}{Z(\beta)} \int_0^\beta g(\lambda) \, \mathrm{Tr}[y^{\beta-\lambda} A^\dagger y^\lambda B] \, d\lambda \tag{3}$$

Two common choices are:

Physical $(A|B)_\beta^P := Z^{-1} \, \mathrm{Tr}[\rho A^\dagger B] + (A \leftrightarrow B)$

Wightman $(A|B)_\beta^W := Z^{-1} \, \mathrm{Tr}[\rho^{1/2} A^\dagger \rho^{1/2} B] + (A \leftrightarrow B)$

In the limit $T \to \infty$ or $\beta \to 0$, these all give the Hilbert-Schmidt inner product.


Quantum Mechanics
Hydrodynamics
Operator
Growth
Hypothesis
Complexity
Thermalization
Chaos

Summary: arXiv: 1812.08657

- The hypothesis governs operator growth in chaotic, closed quantum systems

$$b_n = \begin{cases} A\frac{n}{W(n)} + O(1) \sim \frac{An}{\log n} & \text{if } d = 1 \\ \alpha n + \gamma + o(1) \sim \alpha n & \text{if } d \geq 2 \end{cases}$$

- Emergence of hydrodynamics in a computationally tractable scheme.
- The **operator growth rate** α also controls the growth of complexity and chaos in quantum systems: $\lambda_L \leq 2\alpha$

SYK-q	2	3	4	7	10	∞
$2\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
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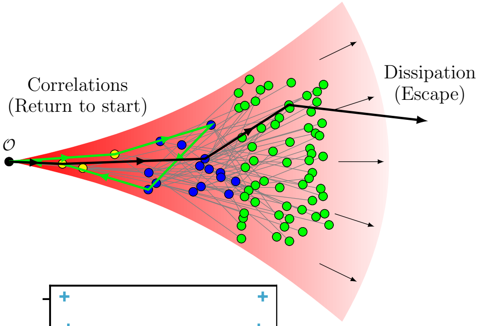
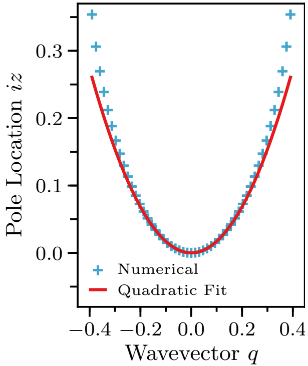




Future Work

- Log corrections disrupt asymptotics in 1D. How does our numerical technique still work in 1D?
- Can we prove the hypothesis within random matrix theory?
- How can we extend the hypothesis to finite temperature?
- Can we compute b_n in QMC or other numerical techniques in 2D?
- Can we say anything about the MBL transition with this notion of chaos/ergodicity?
- Can we measure α experimentally? Perhaps from $\Phi(\omega)$ at large ω?

Extra Slides

Translation-Invariant MBL (Preliminary)

Ising MBL model:

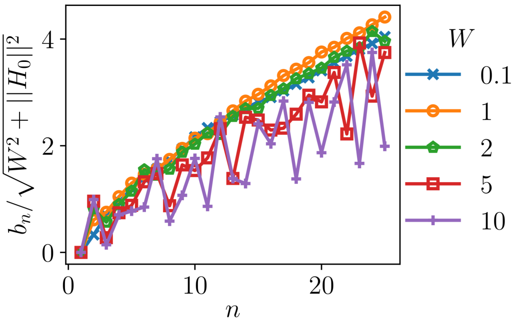
$$H = \sum_j Z_j Z_{j+1} + 1.05 X_j + W_j Z_j.$$

To recover translation-invariance, promote

$$W_j \in \{-1, 1\} \to W \hat{\tau}_j^z$$

where $\hat{\tau}_j^z$ is a "binary disorder operator".

$$\widetilde{H} = \sum_j Z_j Z_{j+1} + 1.05 X_j + W \hat{\tau}_j^z Z_j.$$



Pro: can compute Lanczos *directly in the thermodynamic limit*.
Con: doubled on-site operator space.

Translation-Invariant ℓ-bits (Preliminary)

Start with $\mathcal{O} = Z$.

1. Truncate in Krylov space:

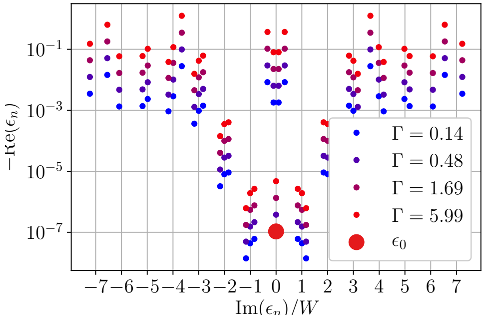
$$T(\Gamma) := \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & b_N \\ \vdots & \vdots & \vdots & b_N & -i\Gamma \end{pmatrix}.$$

2. Solve

$$T\mathcal{A}_\alpha = \varepsilon_\alpha \mathcal{A}_\alpha$$

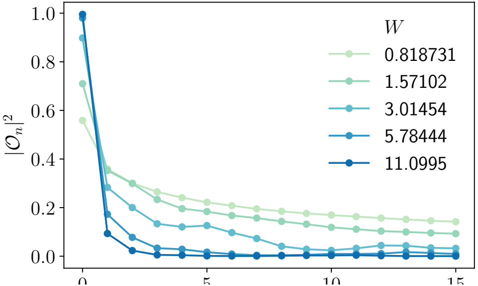
3. Smallest eigenvalue $\varepsilon_0(\Gamma) = 0 - iE_0\Gamma$ with $\mathcal{A}_0$ well-localized to small n.

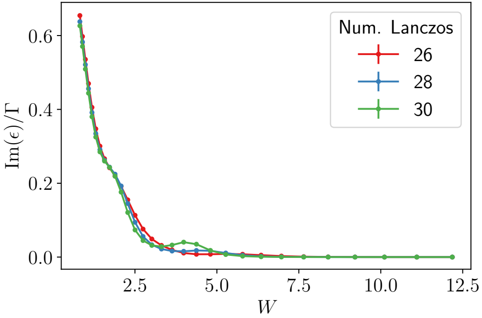
Interpretation: $\mathcal{A}_0$ is a "translation- and disorder-averaged ℓ-bit".



MBL Transition (Preliminary)

- Can we see the MBL transition?
- For large-W, the "ℓ-bit" is exponentially localized in the chain.
- For small-W, it decays as a power-law (fairly generic in ETH systems).
- Interpretation: Lanczos gives a non-perturbative probe of the MBL transition.

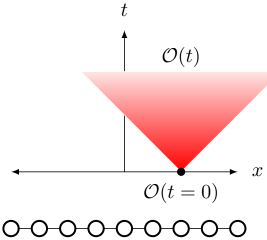




History

Mathematical Results

- Araki (1969)
- Lieb-Robinson Bound (1972)
- ETH (1994)
- ADHH Theorem (2015)



OTOCs

- Quantum version of Lyapunov exponent (Kitaev)
- Maldacena-Shenkar-Stanford bound at low-T (2015)
- Computable in SYK, large-N, holography...
- Only well-defined semiclassically

Random Unitaries

- Solvable models of quantum chaos
- Local, finite-N, operator front propagation
- Emergent dissipation
- Non-Hamiltonian dynamics, no Lyapunov exponents
- (Nahum, Khemani, Huse, Pollmann, etc)

The Lanczos Algorithm

The Lanczos algorithm iteratively *tridiagonalizes* a matrix

Algorithm:

1. Define

$$|\mathcal{O}_0) := \mathcal{O}, b_0 := 0$$

2. For each n, apply $\mathcal{L}$ to make a new operator:

$$|A_n) := \mathcal{L}\,|\mathcal{O}_{n-1}) - b_{n-1}\,|\mathcal{O}_{n-2})$$

3. Orthogonalize again previous operator:

$$|\mathcal{O}_n) := b_n^{-1}\,|A_n)\,, b_n := (A_n|A_n)^{1/2}$$

4. Repeat until $|\mathcal{O}_n)$ vanishes.

The Liouvillian becomes **tridiagonal**

$$L_{nm} := (\mathcal{O}_n|\mathcal{L}|\mathcal{O}_m) = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

The b_n's are called **Lanczos coefficients** and the $|\mathcal{O}_n)$'s are called **Krylov vectors**.

Log Corrections in 1D

Theorem (Araki 1969) For any Hamiltonian with local interactions

$$C(t+i\tau) = \langle \mathcal{O} | e^{i\mathcal{L}(t+i\tau)} | \mathcal{O} \rangle$$

is an entire function of $t + i\tau \in \mathbb{C}$.

Corollary The asymptotic growth of the Lanczos coefficients is strictly sublinear in one dimension. In fact,

$$b_n \leq A \frac{n}{W(n)}$$

where W is the product-log function defined by $z = W(ze^z)$ whose asymptotic is $W(n) \sim \ln n - \ln \ln n + O(1)$.

Therefore the hypothesis is modified in 1D. We still permit $b_n \geq n^\alpha$ for any $\alpha < 1$.

1D correction via Alex Avdoshkin; Araki 1969; Abanin, De Roeck, Huveneers 2015.

Higher Dimensions

Theorem (Bouch 2011) For $d = 2$ (and higher), there exists a local Hamiltonian whose correlation function fail to be entire. Namely

$$H = \sum_{(x,y)\in\mathbb{Z}^2} Z_{x,y}X_{x+1,y} + X_{x,y}Z_{x,y-1}$$

with $\mathcal{O} = X_{0,0}$. (This achieves linear growth of b_n.)

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Corollary For $d \geq 2$, linear growth $b_n = \alpha n + O(1)$ is a tight upper bound for the growth of the Lanczos coefficients.

So the hypothesis survives unscathed in higher dimensions.

1D correction via Alex Avdoshkin & PhD thesis of G.D. Bouch, 2011

ETH Interpretation

Eigenstate thermalization hypothesis:

$$\mathcal{O}_{\alpha\beta} = \mathcal{O}(\overline{E})\delta_{\alpha\beta} + e^{-S(\overline{E})} f_{\mathcal{O}}(\overline{E}, \omega) R_{\alpha\beta} \quad (4)$$

where O is a local observable, $\overline{E} = (E_\alpha + E_\beta)/2$, $\omega = E_\alpha - E_\beta$, $S(\overline{E})$ is the thermodynamic entropy, $R_{\alpha\beta}$ a random variable and $\mathcal{O}(\overline{O})$ and $f_{\mathcal{O}}$ are smooth.

The operator growth hypothesis implies (at $T = \infty$)

$$\text{quantum chaos} \Longleftrightarrow \int d\overline{E}\ f_{\mathcal{O}}(\overline{E}, \omega) = e^{-\frac{\pi|\omega|}{2\alpha} + O(1)}.$$

Finite Temperature

At $T < \infty$ there is a *physical* choice of innner product.
Suppose g is an even measure on the thermal circle:

1. $g : [0, \beta] \to \mathbb{R}$ (or a distribution)
2. $g(\lambda) = g(\beta - \lambda)$
3. $\int_0^\beta g(\lambda) = 1$.

Then, with $y := e^{-H}$, there is a g-inner product:

$$(A|B)_\beta^g := \frac{1}{Z(\beta)} \int_0^\beta g(\lambda) \, \mathrm{Tr}[y^{\beta-\lambda} A^\dagger y^\lambda B] \, d\lambda \tag{5}$$

Two common choices are:

Physical $(A|B)_\beta^P := Z^{-1} \, \mathrm{Tr}[\rho A^\dagger B] + (A \leftrightarrow B)$

Wightman $(A|B)_\beta^W := Z^{-1} \, \mathrm{Tr}[\rho^{1/2} A^\dagger \rho^{1/2} B] + (A \leftrightarrow B)$

In the limit $T \to \infty$ or $\beta \to 0$, these all give the Hilbert-Schmidt inner product.