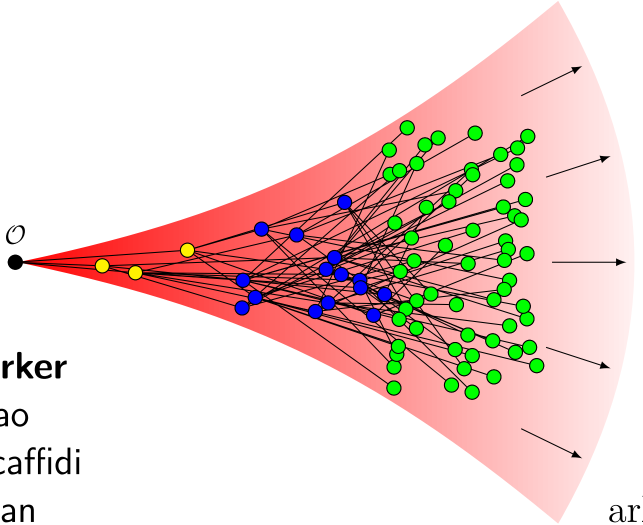
A Universal Operator Growth Hypothesis



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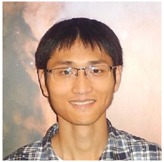
arXiv:1812.08657

Acknowledgements

Collaborators



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Advisor

Joel Moore

Quantum Mechanics

Microscopic description of the system.
Example: Chaotic Ising Model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i$$

Correlation functions:

$$C(t) = \langle \mathcal{O}(t,x)\mathcal{O}(0)\rangle$$

Hard Solution: Hamiltonian dynamics

$$\mathcal{O}(t) = e^{-iHt}\mathcal{O}e^{iHt}.$$

Exact and reversible dynamics.

Hydrodynamics

Macroscopic description of quantum systems as classical PDEs.

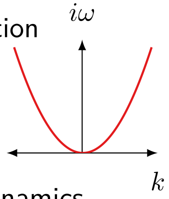
Example: Diffusion of energy

$$\frac{\partial}{\partial t}\varepsilon(t,x) = D\nabla^2\varepsilon(t,x) + \nabla f,$$

with D diffusion, f thermal noise.

Easy Solution: Green's function

$$G(i\omega,k) = \frac{1}{i\omega + Dq^2}$$

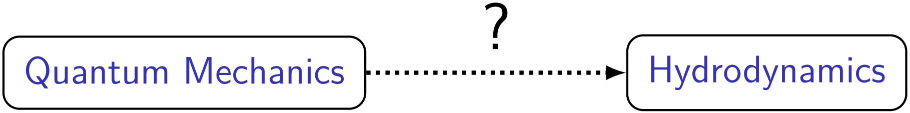


Approximate & irreversible dynamics.

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.59 Z_i$$

$$\frac{\partial}{\partial t}\varepsilon(t,x) = D\nabla^2\varepsilon(t,x) + \nabla f$$

$$D = ?$$



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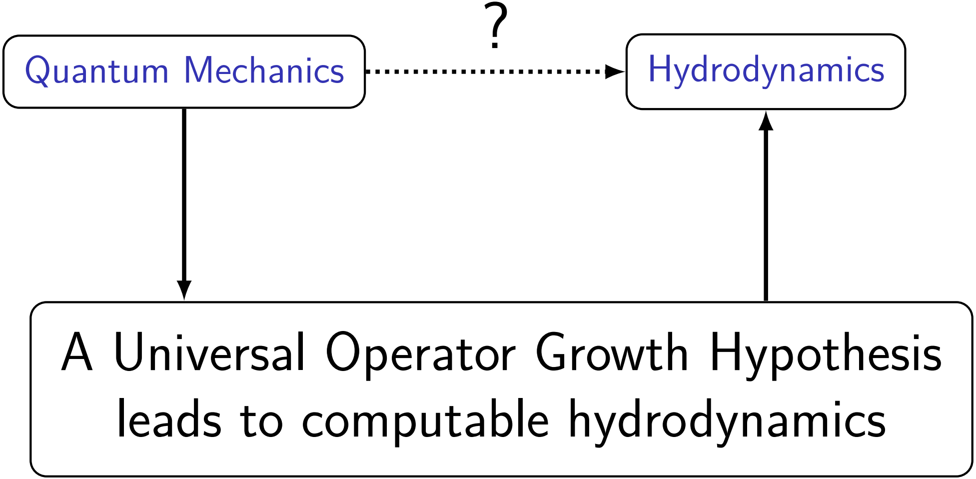

The Graph of Operators

Example: Chaotic Ising Model

$$H = \sum_i X_i + 1.05 Z_i Z_{i+1} + 0.5 Z_i.$$

Problem: Compute $C(t) = \langle \mathcal{O}(t)\mathcal{O}(0) \rangle$.

$$\begin{aligned} \mathcal{O}(t) &= e^{-iHt}\mathcal{O}e^{iHt} \\ &= \mathcal{O} - it[H, \mathcal{O}] + (-it)^2[H, [H, \mathcal{O}]] + \cdots \end{aligned}$$

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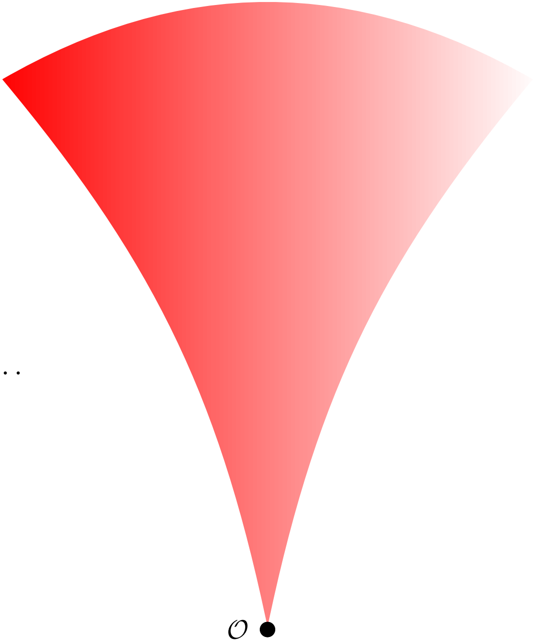
Let's compute!

$$\mathcal{O} = X_1$$



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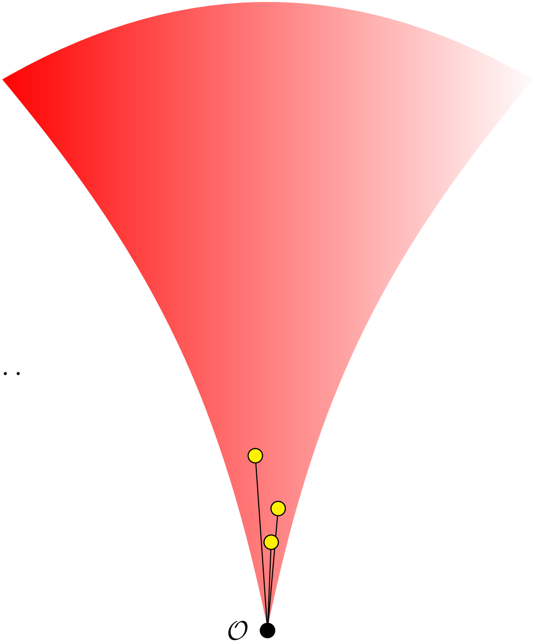
$$\begin{aligned} \mathcal{O} &= X_1 \\ [H, \mathcal{O}] &= 1.05 i Y_1 Z_2 + 1.05 i Z_1 Y_2 + 0.5 i Y_1 \end{aligned}$$



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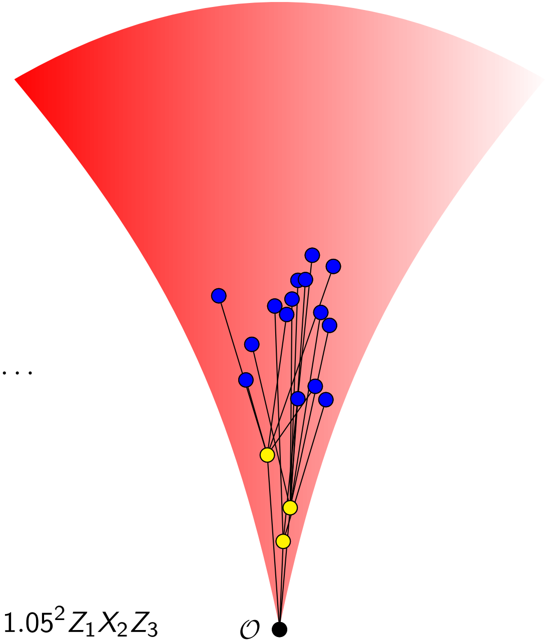
Let's compute!

$$\begin{aligned} \mathcal{O} &= X_1 \\ [H,\mathcal{O}] &= 1.05iY_1Z_2 + 1.05iZ_1Y_2 + 0.5iY_1 \\ [H,[H,\mathcal{O}]] &= 2.1Z_1Z_2 - 2.1Y_1Y_2 \\ &\quad + 1.05^2 Z_0X_1Z_2 + 1.05^2 X_1 + 1.05^2 X_2 + 1.05^2 Z_1X_2Z_3 \end{aligned}$$



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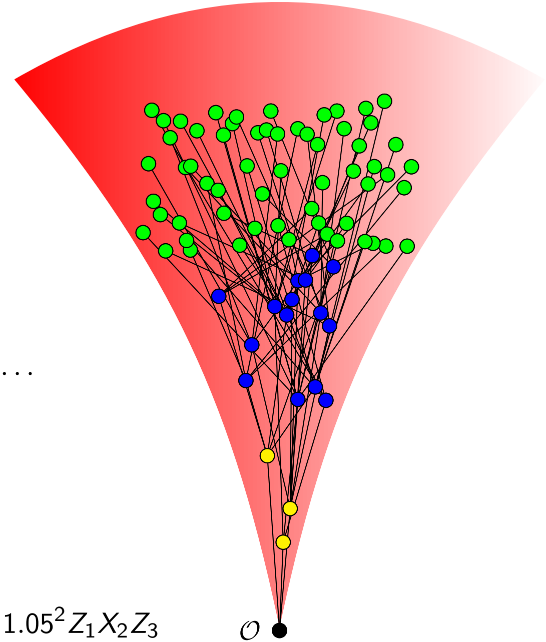

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Let's compute!

$$\mathcal{O} = X_1$$
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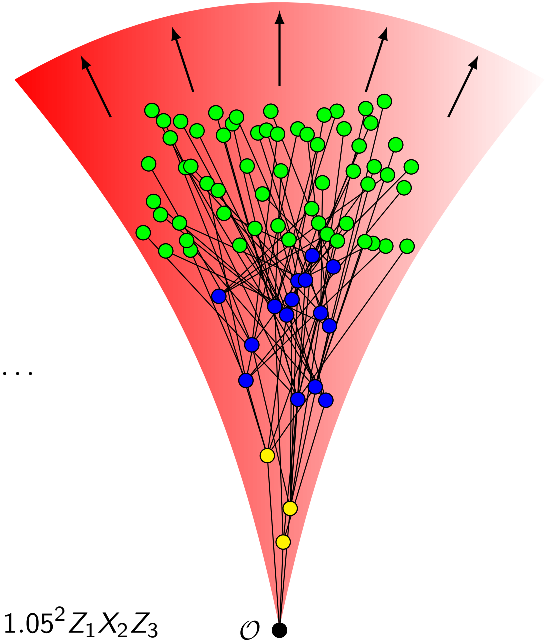
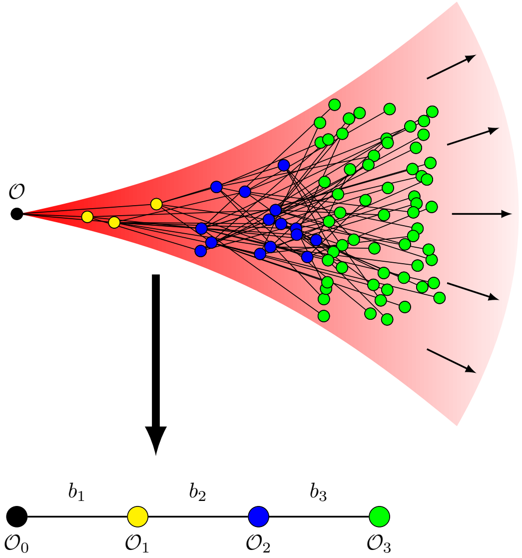
$$[H,[H,\mathcal{O}]] = 2.1Z_1Z_2 - 2.1Y_1Y_2 + 1.05^2 Z_0X_1Z_2 + 1.05^2X_1 + 1.05^2X_2 + 1.05^2Z_1X_2Z_3$$



Recusion Method

Idea: There are so many operators they act as a *thermodynamic bath*. Dynamics should be *universal*.



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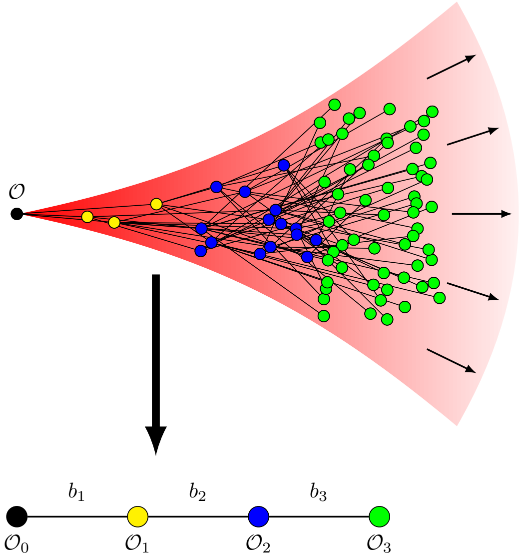
Idea: There are so many operators they act as a *thermodynamic bath*. Dynamics should be *universal*.

Recursion Method: Restate problem as 1d tight-binding model.

$$C(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \varphi_0(t)$$

$$i\partial_t \varphi_n = L_{nm}\varphi_m$$

$$L_{nm} = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$



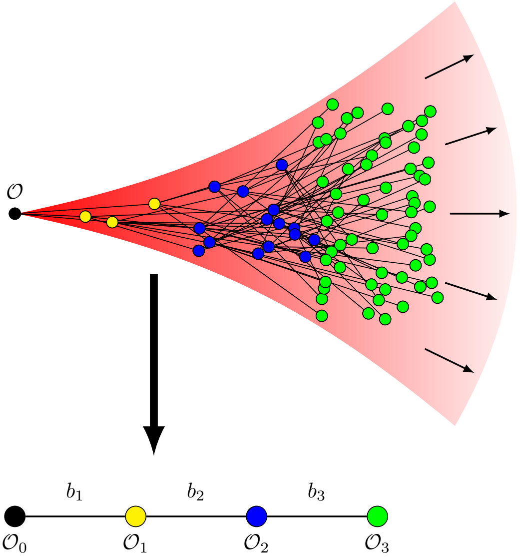
The b_n's are called **Lanczos Coefficients**.

Krylov Vectors

- The Liouvillian is $\mathcal{L} := [H, \cdot]$, and $\mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0)$.
- Take the sequence $\{\mathcal{O}, \mathcal{L}\mathcal{O}, \mathcal{L}^2\mathcal{O}, \dots\}$ and apply Gram-Schmidt to produce an orthogonal basis $\{\mathcal{O}_0 = \mathcal{O}, \mathcal{O}_1, \mathcal{O}_2, \dots\}$.
- The Liouvillian is tridiagonal in this basis

$$L_{nm} := \mathrm{Tr}[\mathcal{O}_n^\dagger \mathcal{L} O_m] = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix} .$$

- 1D Wavefunction defined by $\varphi_n(t) := \mathrm{Tr}[\mathcal{O}_n^\dagger \mathcal{O}(t)]$.
- The b_n's are called **Lanczos coefficients** and the $\mathcal{O}_n$'s are called **Krylov vectors**.



Viswanath & Müller, *The Recursion Method*, 2008.

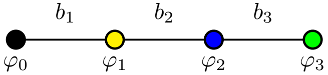
Recusion Method

Recursion Method: Restate problem as 1d tight-binding model.

$$C(t) = \langle \mathcal{O}(t)\mathcal{O}(0)\rangle = \varphi_0(t)$$

$$i\partial_t \varphi_n = L_{nm}\varphi_m$$

$$L_{nm} = \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$



- Exact restatement of $C(t)$ in terms of **Lanczos coefficients** $\{b_n\}_{n=1}^{\infty}$.
- Old method, dating back to the 1980s. See:
  - D.C. Mattis. 1981
  - Viswanath & Müller, *The Recursion Method*, 2008.
- We can compute a few dozen b_n's through numerics.
- "Classification" of dynamics.

Asymptotic	**Growth Rate**	**System Type**
$b_n \sim O(1)$	constant	Free models
$b_n \sim O(\sqrt{n})$	square-root	Integrable models
$b_n \sim$???	???	Chaotic models
$b_n \gtrsim O(n)$	superlinear	Disallowed

Chaotic Examples

$$H_1 = \sum_i X_i X_{i+1} + 0.709 Z_i + 0.9045 X_i$$

$$H_2 = H_1 + \sum_i 0.2 Y_i$$

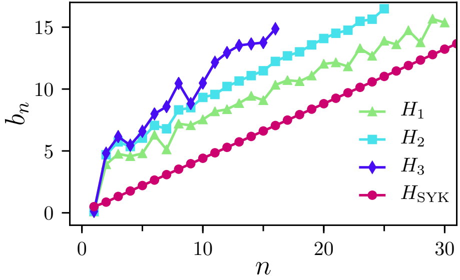
$$H_3 = H_1 + \sum_i 0.2 Z_i Z_{i+1}$$

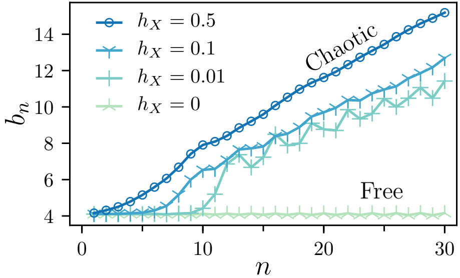
$$H(h_X) = \sum_i X_i X_{i+1} - 1.05 Z_i + h_x X_i$$

$$H_{\mathrm{SYK}}^{(q)} = i^{q/2} \sum_{1 \le i_1 < i_2 < \cdots < i_q \le N} J_{i_1 \dots i_q} \gamma_{i_1} \cdots \gamma_{i_q},$$

$$\overline{J^2_{i_1 \dots i_q}} = 0,$$

$$\overline{J^2_{i_1 \dots i_q}}^2 = \frac{(q-1)!}{N^{q-1}} J^2$$

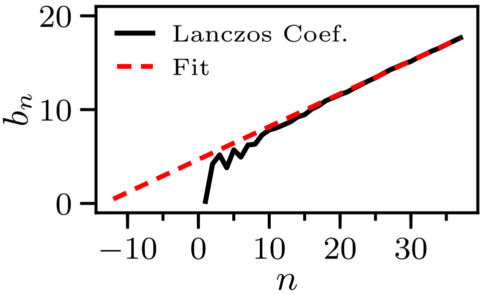




Hypothesis: In a chaotic quantum system, the Lanczos coefficients b_n are asymptotically linear, i.e. for $\alpha, \gamma \geq 0$,

$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$

Asymptotic	Growth Rate	System Type
$b_n \sim O(1)$	Constant	Free models
$b_n \sim O(\sqrt{n})$	Square-root	Integrable models
$b_n \sim O(n)$	Linear	Chaotic models
$b_n \gtrsim O(n)$	Superlinear	Disallowed



Exact Asymptotic Behavior

- Consider

$$\widetilde{b}_n := \alpha\sqrt{n(n-1+\eta)} \xrightarrow{n\gg 1} \alpha n + \gamma.$$

- We can solve this exactly:

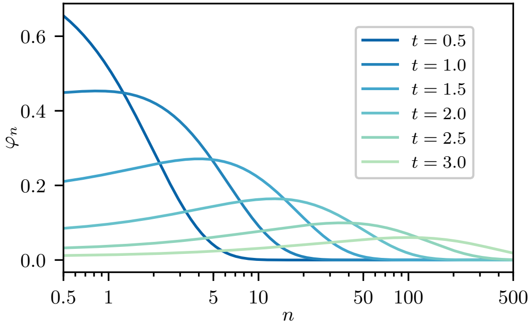
$$\widetilde{\varphi}_n(t) = \sqrt{\frac{(\eta)_n}{n!}}\tanh(\alpha t)^n \operatorname{sech}(\alpha t)^\eta$$

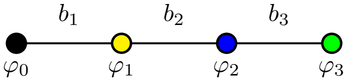
where $(\eta)_n = \eta(\eta+1)\cdots(\eta+n+1)$.

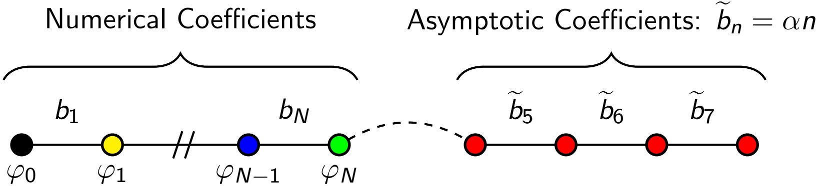
- Expected "position" in the 1D chain is

$$(n(t)) = \eta\sinh(\alpha t)^2 \sim e^{2\alpha t}.$$

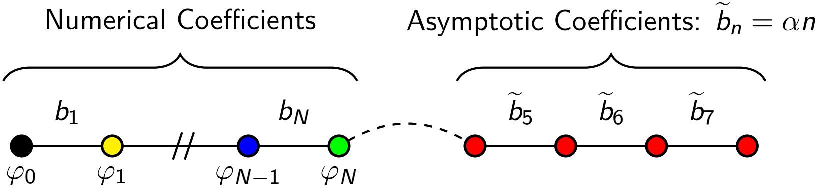
- The wavefunction runs away "irreversibly" into the 1D chain.







Numerical Coefficients
Asymptotic Coefficients: $\widetilde{b}_n = \alpha n$
b_1
b_N
$\widetilde{b}_5$
$\widetilde{b}_6$
$\widetilde{b}_7$
φ_0
φ_1
φ_{N-1}
φ_N



$$L = \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & b_2 & 0 \\ 0 & b_2 & 0 & \ddots \\ 0 & 0 & \ddots & \ddots \end{pmatrix}$$

$$\approx \begin{pmatrix} 0 & b_1 & 0 & 0 \\ b_1 & 0 & \ddots & 0 \\ 0 & \ddots & \ddots & b_N \\ 0 & 0 & b_n & \widetilde{G^{(N)}}(z) \end{pmatrix}$$

$$G(z) = \int dt\, e^{izt} \langle \mathcal{O}(t)\mathcal{O}(0) \rangle$$

$$\approx \cfrac{1}{z - \cfrac{b_1^2}{z - \cfrac{b_2^2}{\cfrac{\ddots}{z - b_N^2 \widetilde{G^{(N)}}(z)}}}}$$

$$\widetilde{G^{(N)}}(z) = \Gamma(N+1)\Gamma\left(\frac{z+1}{2}\right) \times {}_2F_1\left(N+1, \frac{z+1}{2}, \frac{z+2N+3}{2}; -1\right)$$

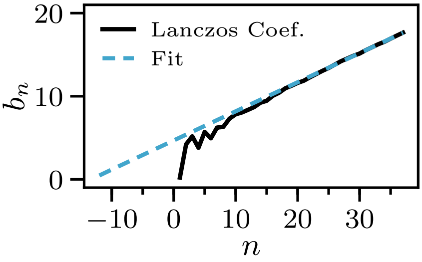
Algorithm

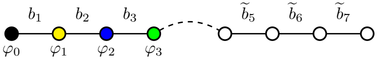
0. Choose a local operator $\mathcal{O}$ whose correlation $C(t) = \mathrm{Tr}[\mathcal{O}(t)\mathcal{O}(0)]$ should be hydrodynamical.

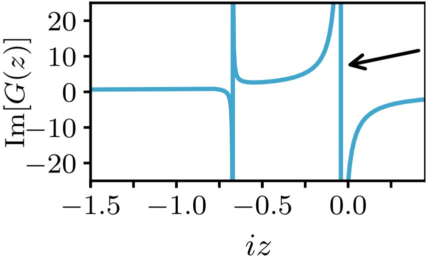
1. Compute $b_1, \ldots, b_N$ via infinite exact diagonalization and fit the slope α.

2. Stitch together the b_n's and the asymptotic solution $\widetilde{G(N)}$.

3. Identify the pole closest to the origin to extract the hydrodynamical dispersion relation.







Diffusion in the Chaotic Ising Model

Chaotic Ising Model

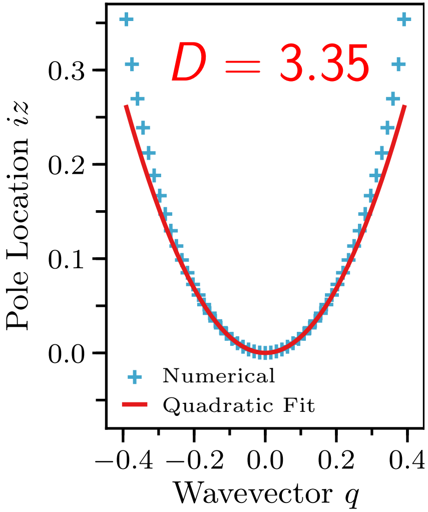
$$H = \sum_j X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j$$

Initial operator at wavevector k:

$$\mathcal{O}_k = \sum_j e^{ikj} \left(X_j + 1.05 Z_j Z_{j+1} + 0.5 Z_j \right)$$

We see the dispersion relation for diffusion

$$\frac{d}{dt}\epsilon(t,x) = D\nabla^2\epsilon(t,x).$$



Summary

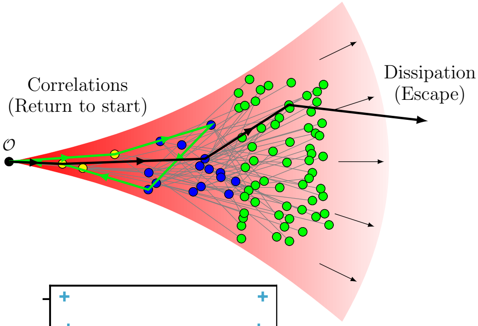
- The hypothesis governs operator growth in chaotic, closed quantum systems

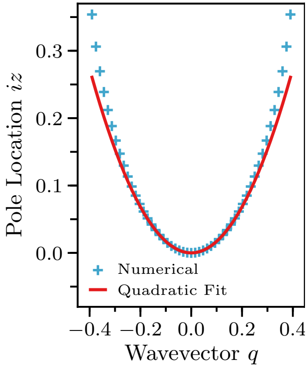
$$b_n \xrightarrow{n \gg 1} \alpha n + \gamma.$$

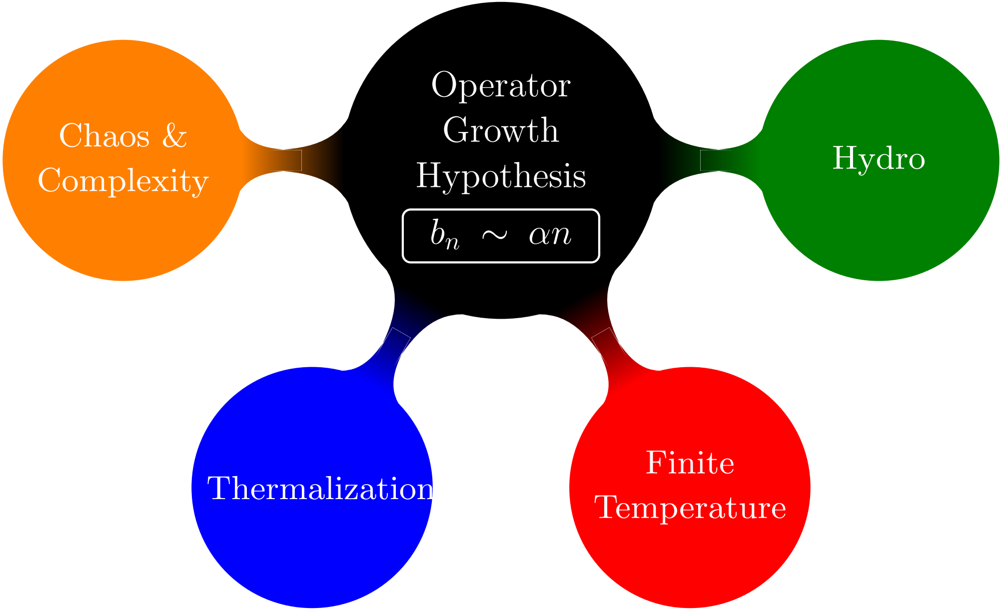
- Emergence of hydrodynamics in a computationally tractable scheme.
- The **operator growth rate** α also controls the growth of complexity and chaos in quantum systems. [Talk in Session P24 by Xiangyu Cao]

SYK-q	2	3	4	7	10	∞
$\alpha/\mathcal{J}$	0	0.461	0.623	0.800	0.863	1
$\lambda_L/(2\mathcal{J})$	0	0.454	0.620	0.799	0.863	1

- arXiv:1812.08657






Operator Growth Hypothesis
$b_n \sim \alpha n$
Chaos & Complexity
Hydro
Thermalization
Finite Temperature

Extra Slide: Other Results

- Other guises of α: relation to the spectral function, analytic structure of $C(t)$, experimental probes.
- The exponential growth of Krylov complexity suggests that 2α can be interpretted as a Lyapunov exponent.
- Formal of complexity: Krylov-complexity and Q-complexities
- Theorem: Krylov complexity grows faster than any other complexity, including operator size and OTOCs.
- The theorem above implies the so-called "quantum bound on chaos" at low temperatures.
- Most of this story carries over directly to the classical case.
- One can show the SYK model obeys the hypothesis directly, and compute most of these quantities exactly.

Extra Slide: Operator Space

We move up a level of abstraction from the space of states to the **space of operators**.

- Operators $\mathcal{O}$ are now "kets", $|\mathcal{O})$.
- e.g. $|\mathcal{O}) = X_1 \otimes Y_2 \otimes Z_3 + 0.3Y_1 \otimes X_2$.
- A basis of operators is the set of **Pauli Strings**

$$|\boldsymbol{\alpha}) = \sigma^{\alpha_1} \otimes \sigma^{\alpha_2} \otimes \cdots \otimes \sigma^{\alpha_n}$$

  for $\alpha_i = 0, 1, 2, 3$.
- Operator inner product:

$$(A|B) := \mathrm{Tr}[A^\dagger B].$$

- The **Liouvillian superoperator** gives the commutator of an operator against the Hamiltonian

$$\mathcal{L} = [H, \cdot].$$

- Heisenburg equation of motion

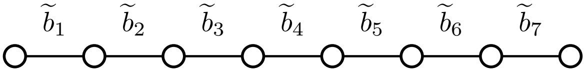
$$-i\frac{d\mathcal{O}}{dt} = [H, \mathcal{O}] \to -i\frac{d\,|\mathcal{O})}{dt} = \mathcal{L}\,|\mathcal{O})\,.$$

- By Baker-Campbell-Hausdorff,

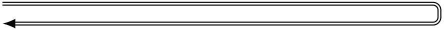
$$\mathcal{O}(t) = e^{iHt}\mathcal{O}e^{-iHt} \to |\mathcal{O}(t)) = e^{-i\mathcal{L}t}\,|\mathcal{O})\,.$$

- Operators evolve in operator space like states in state space.

Extra Slide: The Recursion Method



$$\widetilde{G}(z) = \sum_{\text{paths}}$$



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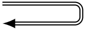
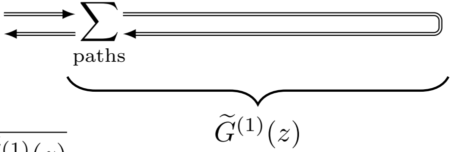


$$+$$



$$= \frac{1}{1 + \widetilde{b}_1^2 \widetilde{G}^{(1)}(z)}$$

$$= \frac{1}{1 + \cfrac{\widetilde{b}_1^2}{1 + \cfrac{\widetilde{b}_2^2}{1 + \widetilde{b}_3^2 \widetilde{G}^{(3)}(z)}}}$$