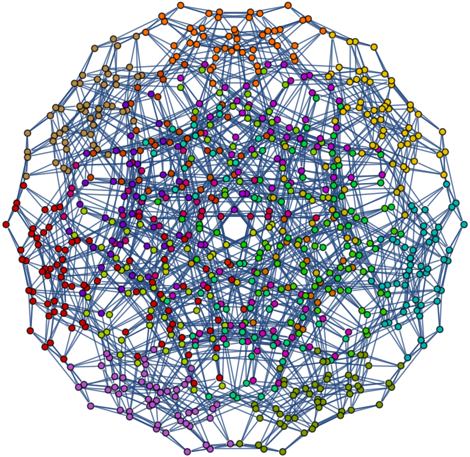
Cluster Algebra Structures for Scattering Amplitudes in $\mathcal{N} = 4$ Super Yang-Mills



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  Prof. Anastasia Volovich, Brown University Physics
- Team: Prof. Marcus Spradlin, Brown University Physics,
  Adam Scherlis, Physics PhD Candidate Stanford University
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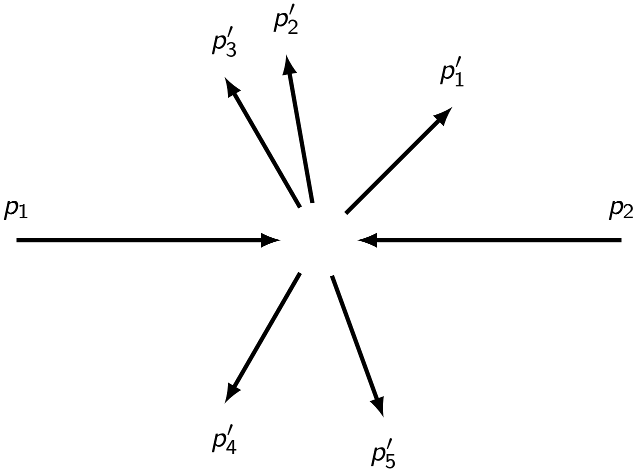
  - arXiv: 1507.01950
  - Journal of High Energy Physics, to appear.
- Dr. John Golden, Prof. Alexander Goncharov
- The amplitudes community.

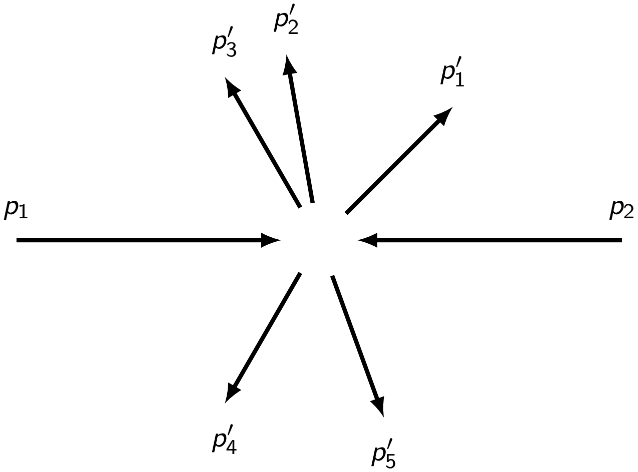
Scattering Amplitudes

Scattering Amplitudes



Scattering Amplitudes

If p_1 and p_2 go in, what is the *probability* of getting $p'_1, \ldots p'_5$ out?



Scattering Amplitudes

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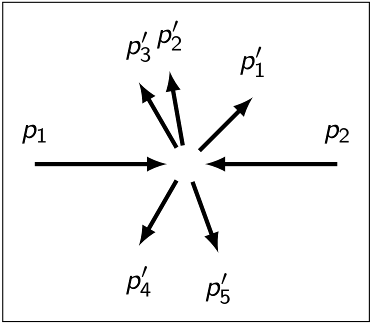

$\Longrightarrow$

Scattering Amplitude

$$\mathcal{A}(p_1, p_2; p'_1, p'_2, p'_3, p'_4, p'_5)$$

$$\mathcal{A} : \mathbb{M}^n \to \mathbb{C}$$

Computing Amplitudes

Quantum Field Theory

↓

Feynman Diagrams

↓

Lots of complicated terms

DDS Formula

$$R^{(2)}_{6,WL}(u_1,u_2,u_3) = \qquad \text{(H.1)}$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)+$$

$$\frac{1}{24}\pi^2 G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)+\frac{1}{24}\pi^2 G\left(\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$

$$\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)+\frac{3}{2}G\left(0,0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)+$$

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$$\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},0,\frac{1}{u_3};1\right)+$$

$$G\left(0,\frac{1}{u_1},0,\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_1},\frac{1}{u_1+u_3};1\right)-$$

$$\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_1},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_1};1\right)+$$

$$G\left(0,\frac{1}{u_2},0,\frac{1}{u_1+u_2};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_2},0,\frac{1}{u_3};1\right)+G\left(0,\frac{1}{u_2},0,\frac{1}{u_2+u_3};1\right)-$$

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$$\frac{1}{2}G\left(0,\frac{1}{u_2},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},0;1\right)-\frac{1}{4}G\left(0,\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)+$$

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$$\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_1};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},0,\frac{1}{u_2};1\right)+G\left(0,\frac{1}{u_3},0,\frac{1}{u_1+u_3};1\right)+$$

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$$\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_1+u_3};1\right)-\frac{1}{2}G\left(0,\frac{1}{u_3},\frac{1}{u_3},\frac{1}{u_2+u_3};1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},0,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},0;1\right)-$$

$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},1;1\right)+\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3},\frac{1}{1-u_3};1\right)-$$

$$\frac{1}{4}G\left(0,\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},0,\frac{1}{1-u_2};1\right)+$$

$$\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},0;1\right)-\frac{1}{4}G\left(0,\frac{u_3-1}{u_2+u_3-1},\frac{1}{1-u_2},1;1\right)+$$

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$$\frac{1}{4}G\left(\frac{1}{1-u_1},1,\frac{1}{u_3},0;1\right)+\frac{1}{2}G\left(\frac{1}{1-u_1},\frac{1}{1-u_1},1,\frac{1}{1-u_1};1\right)+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},0,\frac{1}{1-u_1};1\right)+$$

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$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},1;1\right)-\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{1}{1-u_1},\frac{1}{1-u_1};1\right)-$$

$$\frac{1}{4}G\left(\frac{1}{1-u_1},\frac{u_2-1}{u_1+u_2-1},\frac{u_2-1}{u_1+u_2-1},1;1\right)+$$

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$$\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_2};1\right)-G\left(\frac{1}{u_1},0,0,\frac{1}{u_3};1\right)+\frac{1}{2}G\left(\frac{1}{u_1},0,0,\frac{1}{u_1+u_3};1\right)-$$

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$$\frac{1}{2}G\left(\frac{1}{1-u_2},\frac{1}{1-u_2},1,\frac{1}{1-u_2};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_2},\frac{u_3-1}{u_2+u_3-1},0,1;1\right)-$$

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$$\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_1},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_1+u_2};1\right)-\frac{1}{4}G\left(\frac{1}{u_2},0,\frac{1}{u_2},\frac{1}{u_2+u_3};1\right)-$$

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$$\frac{1}{2}G\left(\frac{1}{1-u_3},\frac{1}{1-u_3},1,\frac{1}{1-u_3};1\right)+\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},0,1;1\right)-$$

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$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},1;1\right)-\frac{79\pi^4}{360}+$$

$$\frac{1}{4}G\left(\frac{1}{1-u_3},\frac{u_1-1}{u_1+u_3-1},\frac{u_1-1}{u_1+u_3-1},\frac{1}{1-u_3};1\right)-G\left(\frac{1}{u_3},0,0,\frac{1}{u_1};1\right)-$$

Plus 17 more pages!

GSVV Formula

$$R_6^{(2)}(u_1,u_2,u_3) = \sum_{i=1}^{3}\left(L_4(x_i^+,x_i^-) + \frac{1}{2}\,\mathrm{Li}_4\left(1-\tfrac{1}{u_i}\right)\right) - \frac{1}{8}\left(\sum_{i=1}^{3}\mathrm{Li}_2\left(1-\frac{1}{u_i}\right)\right)^2 + \frac{1}{24}J^4 + \frac{\pi^2}{12}J^2 + \frac{\pi^4}{72}$$

where

$$L_4(x^+,x^-) = \frac{1}{8!!}\log(x^+x^-)^4 + \sum_{m=0}^{3}\frac{(-1)^m}{(2m)!!}\log(x^+x^-)^m\left(\ell_{4-m}(x^+)+\ell_{4-m}(x^-)\right)$$

as well as

$$\ell_n = \frac{1}{2}\left(\mathrm{Li}_n(x) - (-1)^n\,\mathrm{Li}_n\left(\tfrac{1}{x}\right)\right) \text{ and } J = \sum_{i=1}^{3}\left(\ell_1(x_i^+) - \ell_1(x_i^-)\right).$$

Goncharov, Spradlin, Vergu, Volovich. arXiv: hep-th/1105.1333

Computing Amplitudes

Quantum Field Theory

↓

Feynman Diagrams

↓

Lots of complicated terms

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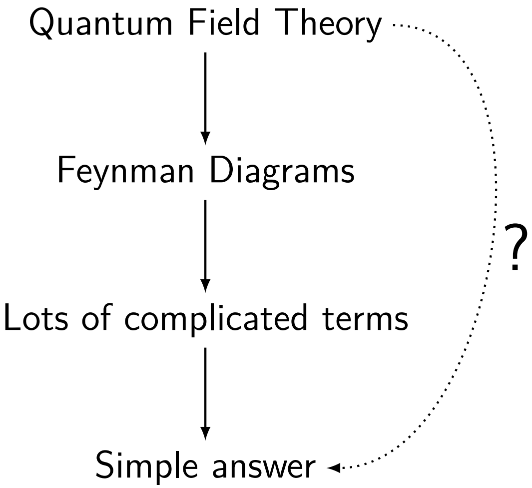
Simple answer

Computing Amplitudes



Theme

Particular scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills can be written as sums of polylogarithms in variables with a cluster algebra structure.

$$\text{“}\mathcal{A}(p_1, \ldots, p_n) = \sum G(x_i, x_j, \ldots, x_k; x_\ell)\text{”}$$

Classical Polylogarithms

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These are *classical* polylogarithms.

Generalized Polylogarithms

Scattering amplitudes involve Goncharov polylogarithms.

The base cases is

$$G(a_1; a_2) = \int_0^{a_2} \frac{dt}{t - a_1}.$$

The rest are defined recursively by

$$G(a_1, \ldots, a_k; a_{k+1}) = \int_0^{a_{k+1}} G(a_2, \ldots, a_k; w_k) \frac{dw_k}{w_k - a_1}$$
$$= \int_0^{a_{k+1}} \int_0^{w_k} \cdots \int_0^{w_2} \frac{dw_1}{w_1 - a_1} \frac{dw_2}{w_2 - a_2} \cdots \frac{dw_n}{w_n - a_k}.$$

These are related by

$$\mathrm{Li}_k(z) = G(\underbrace{0, \ldots, 0}_{k-1}, 1, z).$$

Goncharov. arXiv: math/0103059

Cluster Algebras

Cluster Algebras are a new area of mathematics.

- Discovered in 2002 by Fomin and Zelevinsky.
- A new and exciting field!
- Applications to dozens of areas, including physics.

Fomin, Zelevinsky. arXiv:math/0104151

L. K. Williams, "Cluster algebras: an introduction," arXiv:1212.6263.

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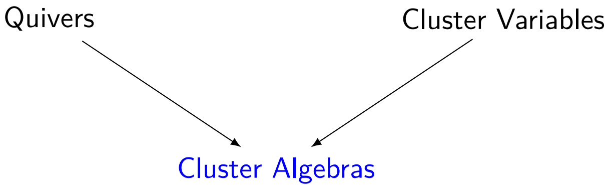

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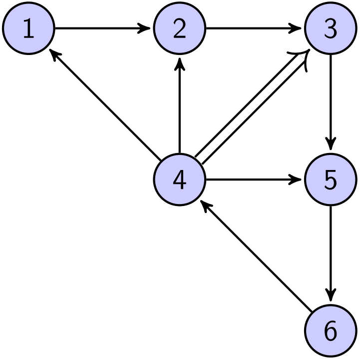
Quivers

Quivers are directed graphs.

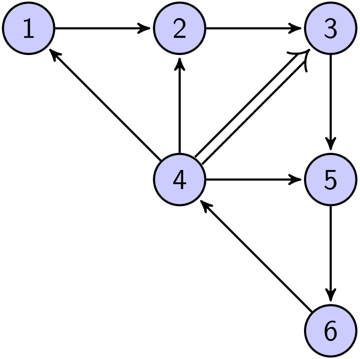


Quivers

Quivers are directed graphs.

Quiver Mutation on node k

1. Complete all triangles $i \to k \to j$.
2. Reverse arrows into or out of k.
3. Remove new 2-cycles.



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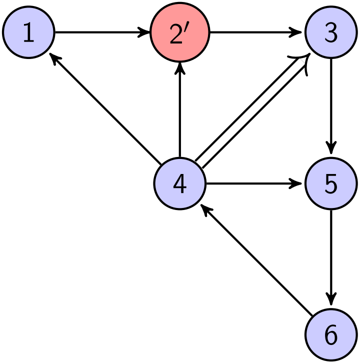

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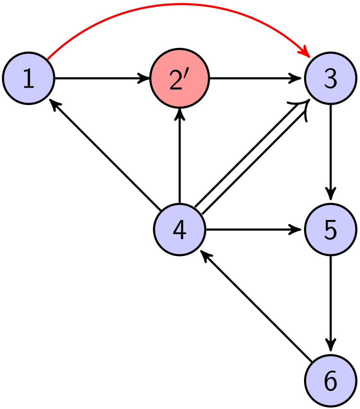

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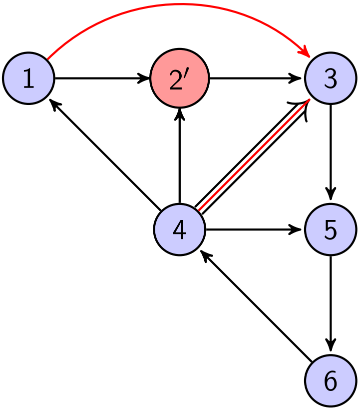

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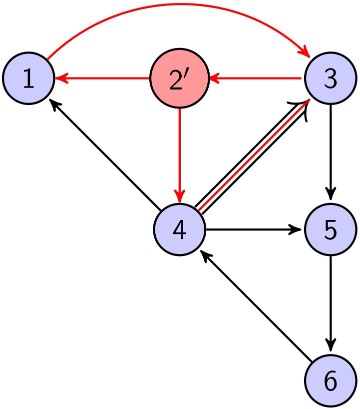
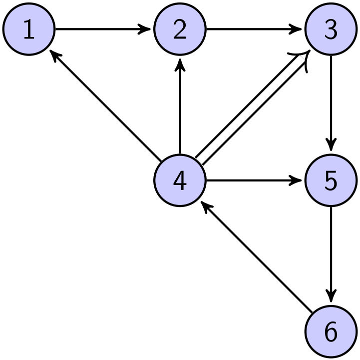


Cluster Variables

Attach **cluster variables** to nodes.



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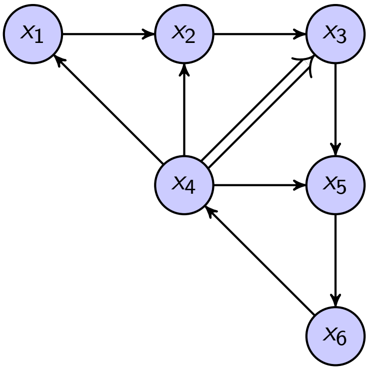

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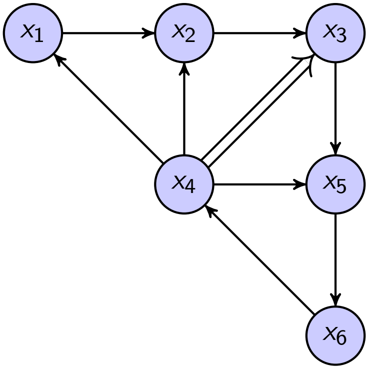
Cluster Variable Mutation on node k:

$$x_i' = \begin{cases} \frac{1}{x_k} & i = k \\ x_i \left(1 + x_k^{\operatorname{sgn} b(i,k)}\right)^{b(i,k)} & i \neq k \end{cases}$$

where

$$b(i,k) = \#\text{arrows from } i \text{ to } k.$$



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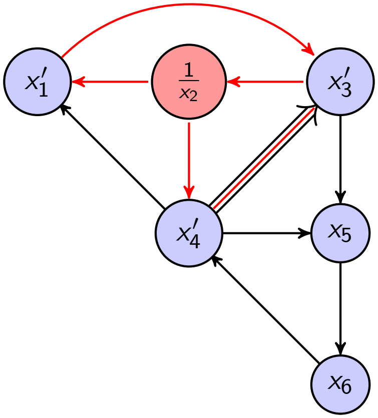

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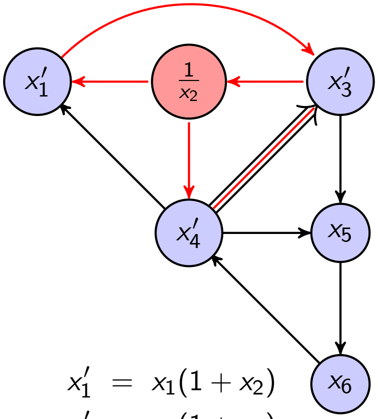

$$\begin{aligned} x_1' &= x_1(1 + x_2) \\ x_3' &= x_3(1 + x_2) \\ x_4' &= x_4\left(1 + x_2^{-1}\right) \end{aligned}$$

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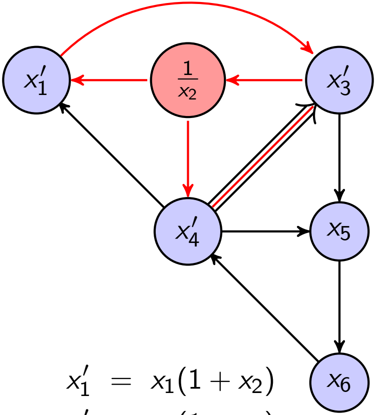

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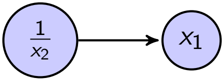
Cluster Algebras

Cluster Algebras are defined iteratively.

1. Choose an initial quiver labelled with cluster variables.
2. Form new clusters by mutating on all nodes.
3. Repeat until no new clusters appear.

The A_2 Cluster Algebra

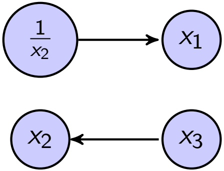
C_1 :



The A_2 Cluster Algebra

C_1 :

C_2 :



$$x_3 = x_1 \left(1 + \left(\frac{1}{x_2}\right)^{-1}\right)^{-1} = \frac{x_1}{1+x_2}$$

The A_2 Cluster Algebra

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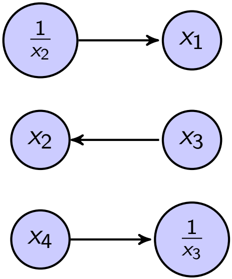
C_2 :

C_3 :



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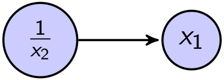
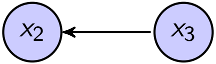


C_2 :



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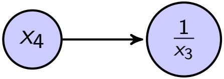
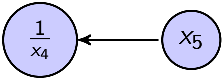


$$x_4 = x_2 \left(1 + x_3^{-1}\right)^{-1} = \frac{x_1 x_2}{1+x_1+x_2}$$

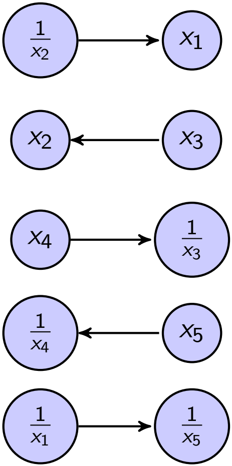
C_4 :



$$x_5 = \frac{1}{x_3} \left(1 + x_4^{-1}\right)^{-1} = \frac{x_2}{1+x_1}$$

The A_2 Cluster Algebra

C_1 : C_2 : C_3 : C_4 : C_5 :



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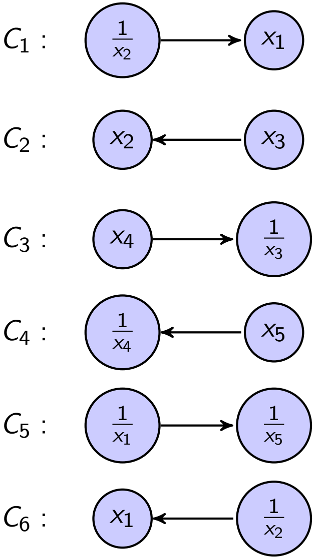

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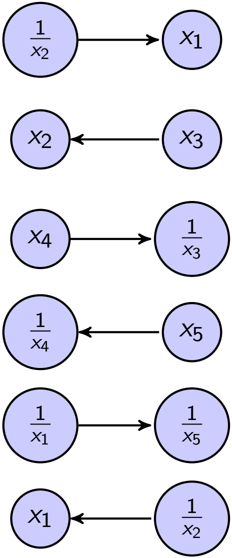

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A_2 Exchange Graph

Exchange graph: join two clusters with an edge if you can mutate between them.

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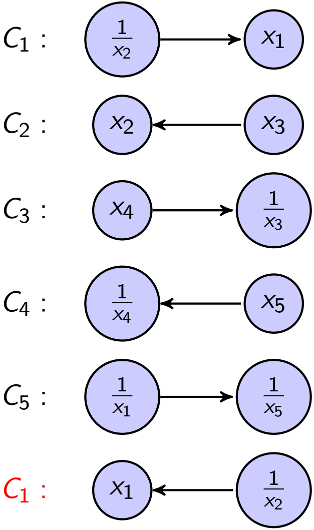

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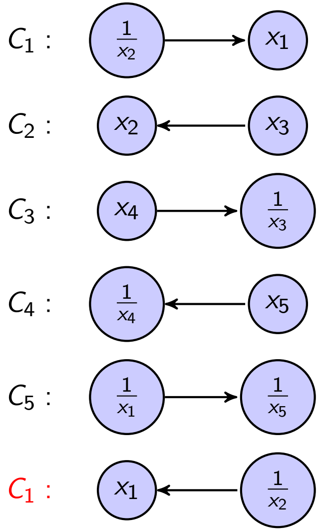
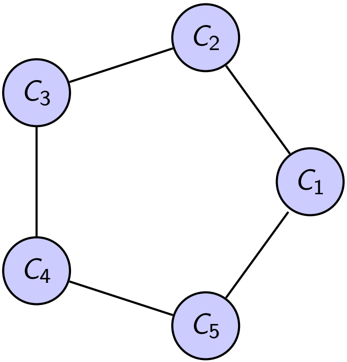
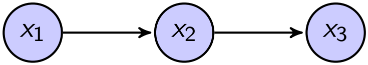




A_3 Exchange Graph



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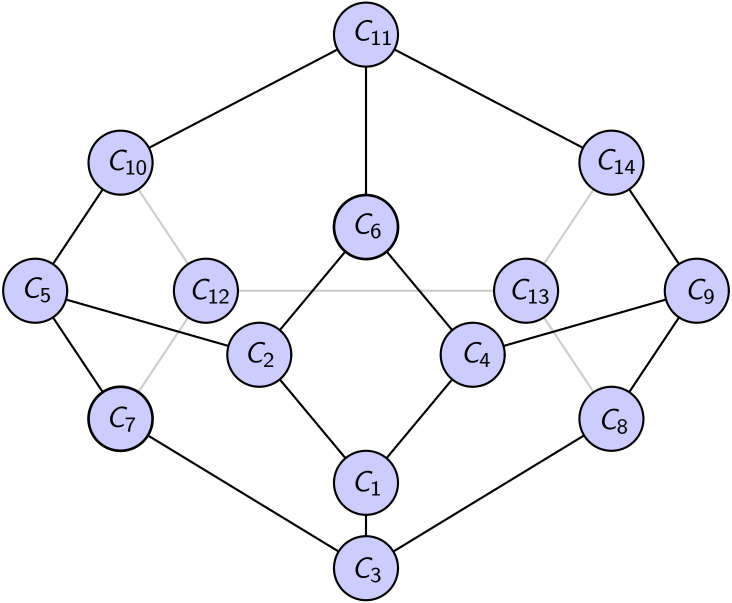
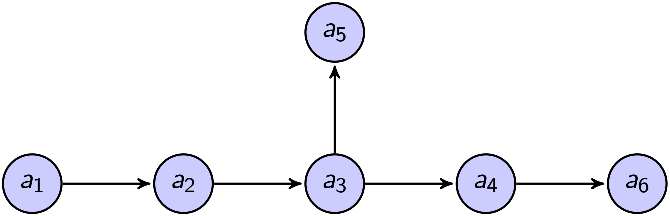


E_6 Exchange Graph



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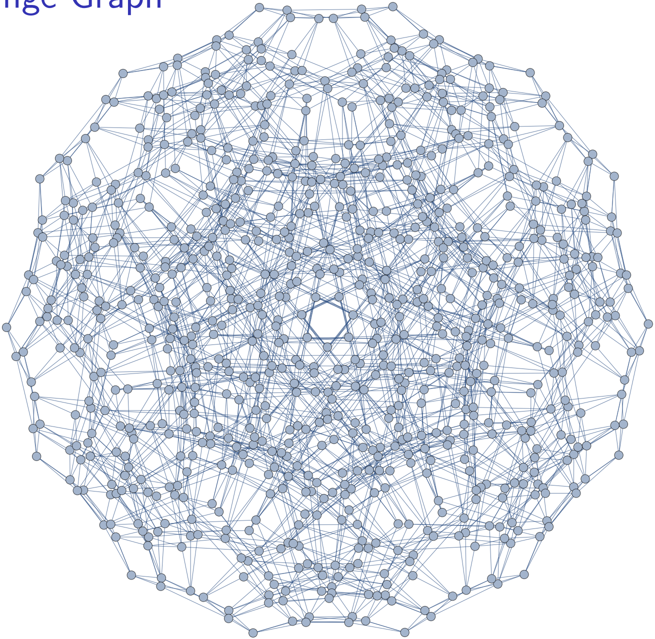

Theme Redux

Particular scattering amplitudes in $\mathcal{N} = 4$ Super Yang-Mills can be written as sums of polylogarithms in variables with a cluster algebra structure.

$$\text{``}\mathcal{A}(p_1, \ldots, p_n) = \sum G(x_i, x_j, \ldots, x_k, x_\ell)\text{''}$$

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How are the x_i's and p_i's related?

The p_i's are 4-momenta, vectors in Minkowski space. Amplitudes are defined on a complicated subvariety of $\mathbb{M}^n$:

$$K_n = \left\{ (p_1, \ldots, p_n) \in \mathbb{M}^n \ : \ \sum_{i=1}^{n} p_i = 0, \ p_k^2 = 0 \quad \forall 1 \leq k \leq n \right\}.$$

Easier to complexify and compactify Minkowski space:

$$K_n \hookrightarrow \mathbf{Gr}(4, n) / \left((\mathbb{C}^*)^{n-1} \right)$$

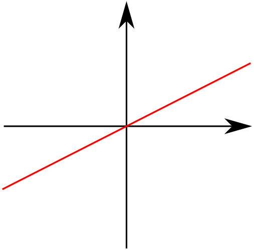
where $\mathbf{Gr}(k, n)$ is the **Grassmannian**: the set of k-plane in (real or complex) n-dimensional space.

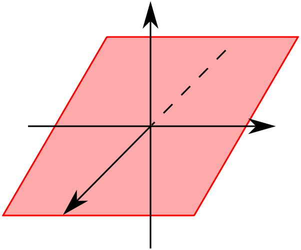
We promote scattering amplitudes to complex-valued functions on this Grassmannian quotient.

Grassmannians

Grassmannians

- $\mathbf{Gr}(1,2)$ is the set of lines through $(0,0)$ in a plane
- $\mathbf{Gr}(2,3)$ is the set of planes through $(0,0,0)$ in 3D space





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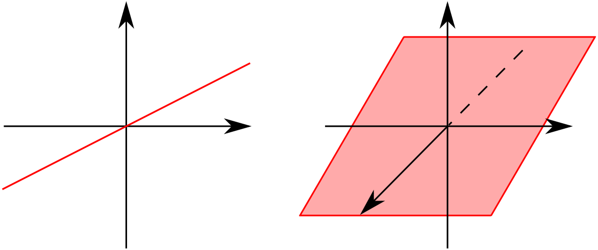

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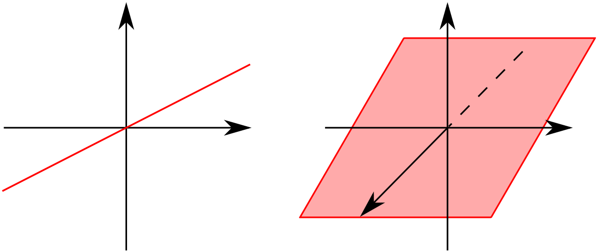

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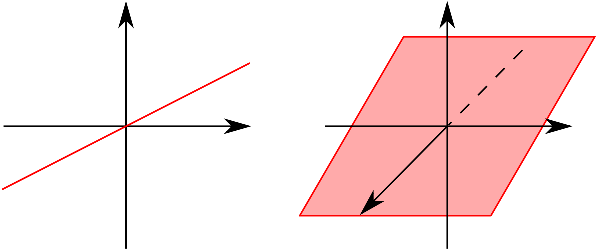
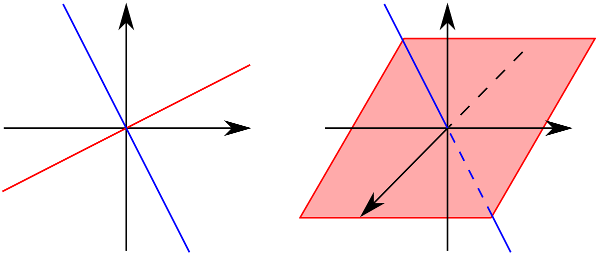


- There is a **cluster algebra** associated with **Gr**(k, n)
- **Cluster variables** describe the orientation of a k-plane
- **Gr**$(1, 2) \cong A_1$: Two clusters $\{m\}, \{1/m\}$

Grassmannians

Grassmannians

- Orthogonal complement: k-planes $\leftrightarrow$ $(n-k)$-planes
- $\mathbf{Gr}(k,n) \cong \mathbf{Gr}(n-k,n)$



- $\mathbf{Gr}(1,2) \cong \mathbf{Gr}(1,2)$
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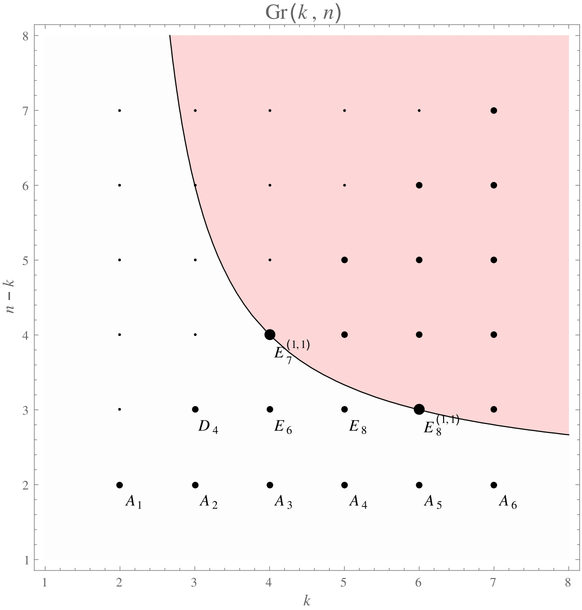
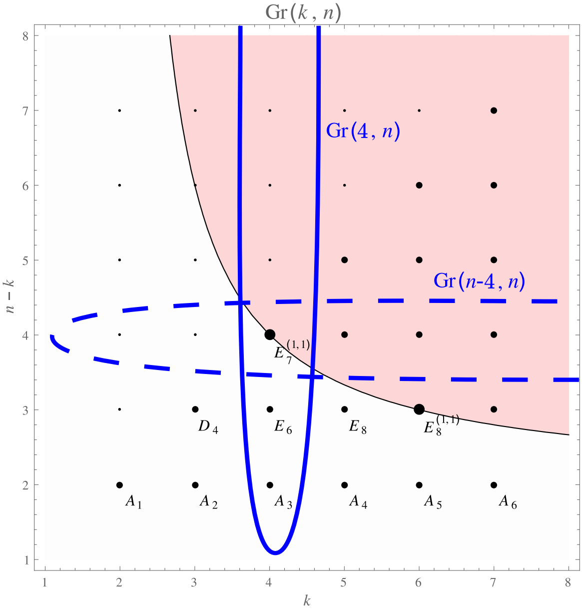
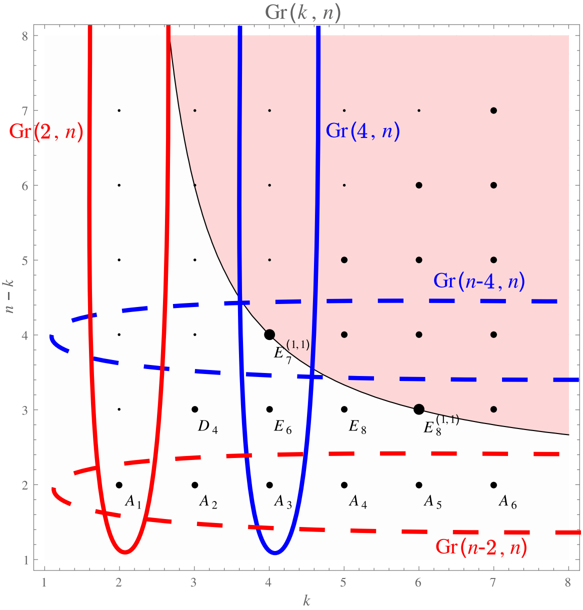
The Cluster Algebra of $\mathbf{Gr}(k, n)$



The Cluster Algebra of $\mathbf{Gr}(4, n)$



The Cluster Algebra of $\mathbf{Gr}(2, n+3) \cong A_n$



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$$\text{``}\mathcal{A}(p_1, \ldots, p_n) = \sum G(\ldots, x_i, \ldots, x_j, \ldots)\text{''}$$

where x_i is a cluster variable for the $\mathbf{Gr}(4, n)$ cluster algebra.

Central Question

Question (physicists): what possible terms can appear?

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Question (mathematicians): what is the structure of the space of **cluster polylogarithm** functions

$$\mathcal{A}_n = \left\{ \mathcal{A} : \mathbf{Gr}(4,n) / \left((\mathbb{C}^*)^{n-1} \right) \xrightarrow{\text{cluster algebra}} \mathbb{C}^n \xrightarrow{\text{polylog}} \mathbb{C} \right\} ?$$

Can we find a “basis”?

Cliques

$$\text{“}\mathcal{A}(p_1, \ldots, p_n) = \sum G(\ldots, x_i, \ldots, x_j, \ldots)\text{”}$$

Which x_i or $\frac{1}{x_i}$ can actually appear? The **rule** is *the difference must factor*:

$$x_i - x_j = \prod_k x_k^{c_k}, \qquad \forall x_i, x_j$$

where $c_k \in \mathbb{Z}$ and k indexes all cluster variables for $\mathbf{Gr}(4, n)$.

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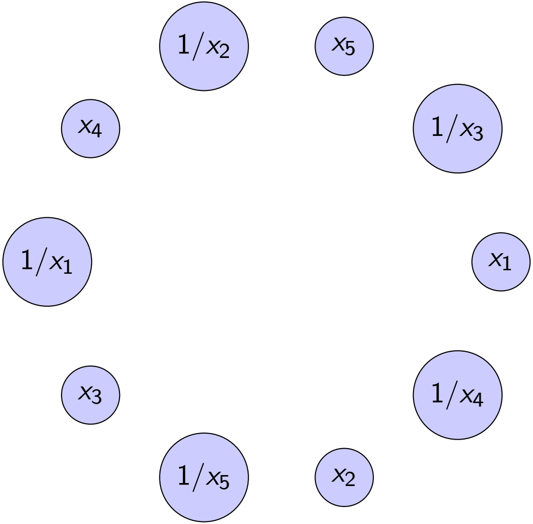
A **clique** is a maximal set of compatible x_i's.

The Factorization Graph

Vertices: all x_i and $y_i = \frac{1}{x_i}$ for a cluster algebra. Connect x_i and x_j if $x_i - x_j$ factors.



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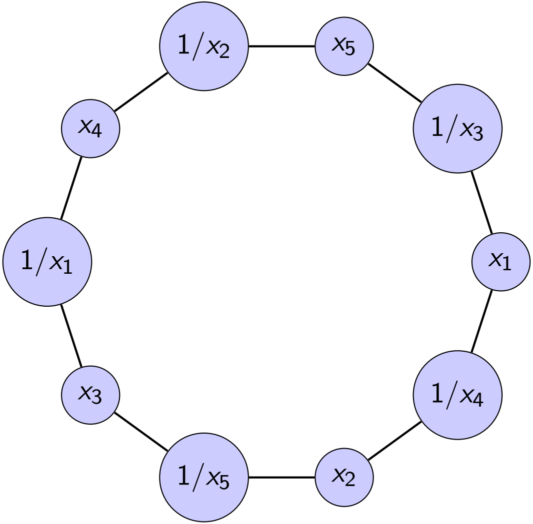
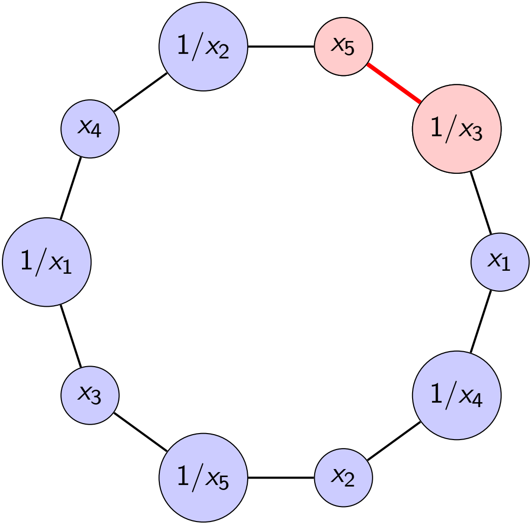


Cliques

What does this mean?



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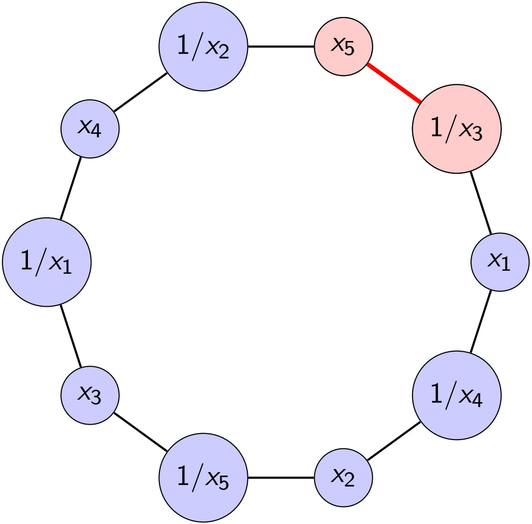

Permitted polylogs:

$G(1/x_3; x_5)$

$G(x_5; 1/x_3)$

$G(1/x_3, 1/x_3; 1/x_3)$

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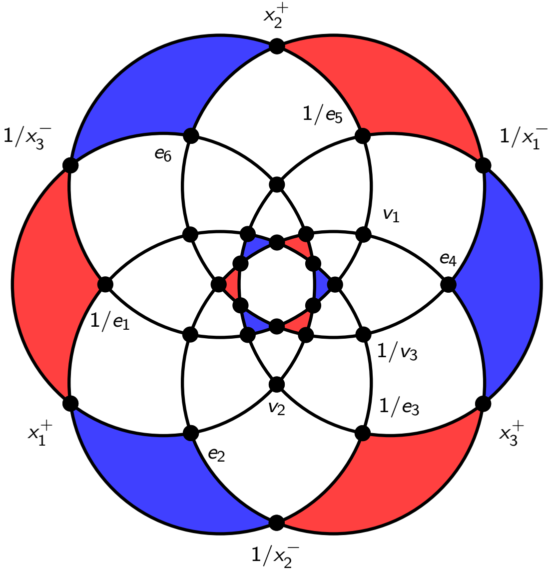
$\vdots$

Cliques for the A_3 Cluster Algebra



Cluster Variables:

$$\left\{x_k^{\pm}, e_k, v_k : 1 \leq k \leq 3\right\}$$

Cliques:

$$\left\{1/x_1^-, x_2^+, 1/e_5\right\}$$
$$\left\{1/x_3^-, x_2^+, e_6\right\}$$
$$\left\{1/x_3^-, x_1^+, 1/e_1\right\}$$

etc.

Brown's Theorem

Francis Brown worked on regular functions over the moduli space of marked points on the Riemann sphere, $\mathfrak{M}_{0,n}$. Correspondance:

$$A_n \text{ cluster algebra } \longleftrightarrow \mathbb{C}[\mathfrak{M}_{0,n+3}]$$

Francis Brown. Multiple zeta values and periods of moduli spaces $\mathfrak{M}_{0,n}$. arXiv: math/0606419

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We can "translate" his work into the language of cluster algebras.

Theorem (Brown): If there is a clique of size n in the A_n cluster algebra, then there exists a minimal algebraic generating set for the space of cluster polylogarithms $\mathcal{A}$.

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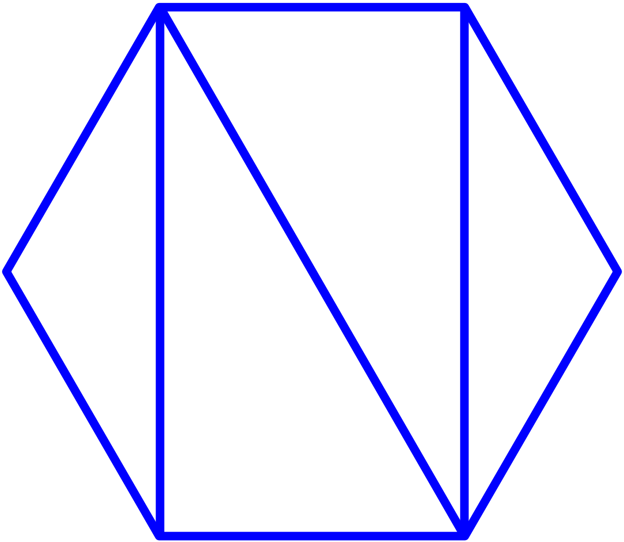
Clique $\longrightarrow$ "Basis" $\longrightarrow$ Cluster Polylogs $\longrightarrow$ Scattering Amplitudes

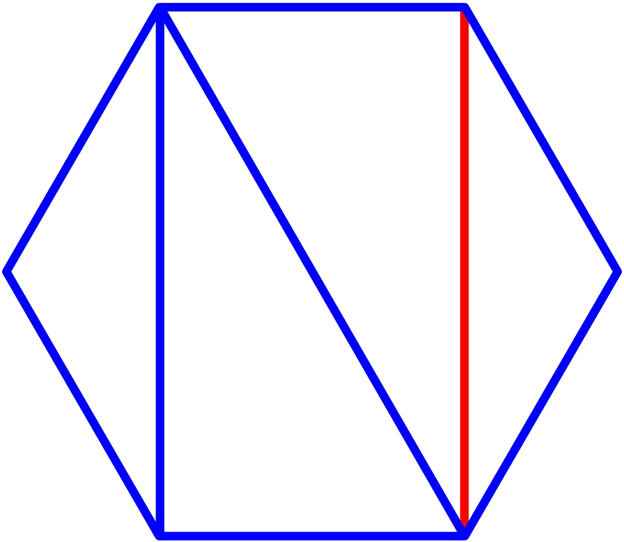
All we need is one n-clique!

Francis Brown. Multiple zeta values and periods of moduli spaces $\mathfrak{M}_{0,n}$. arXiv: math/0606419

Triangulations



Triangulations



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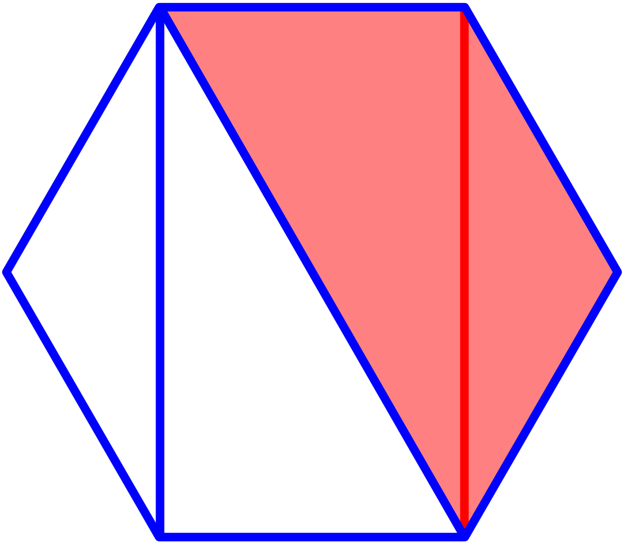

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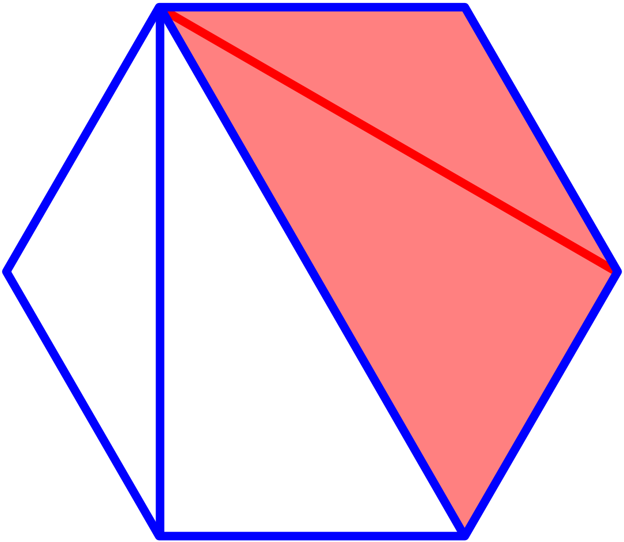

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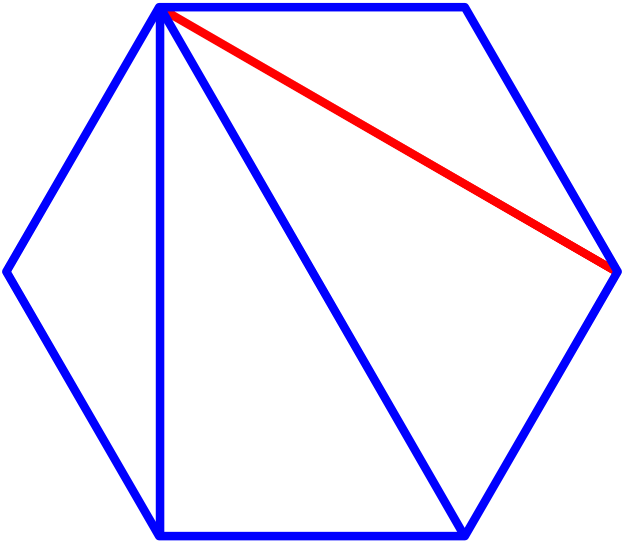

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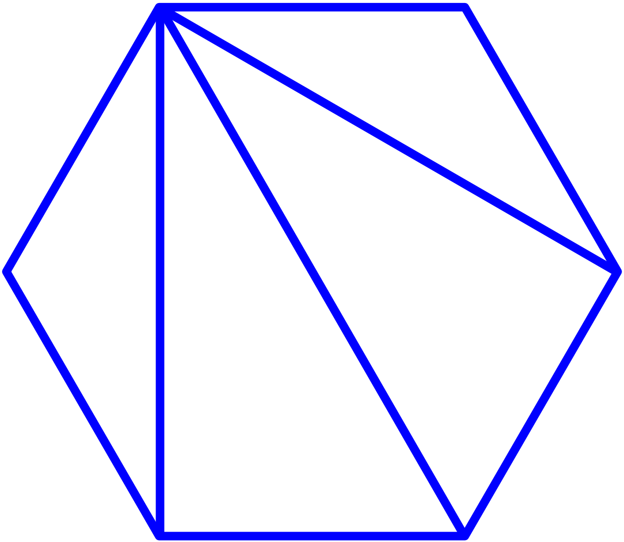

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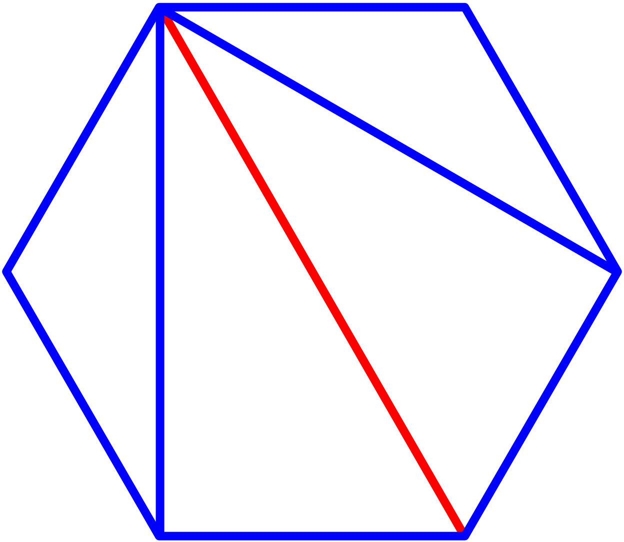

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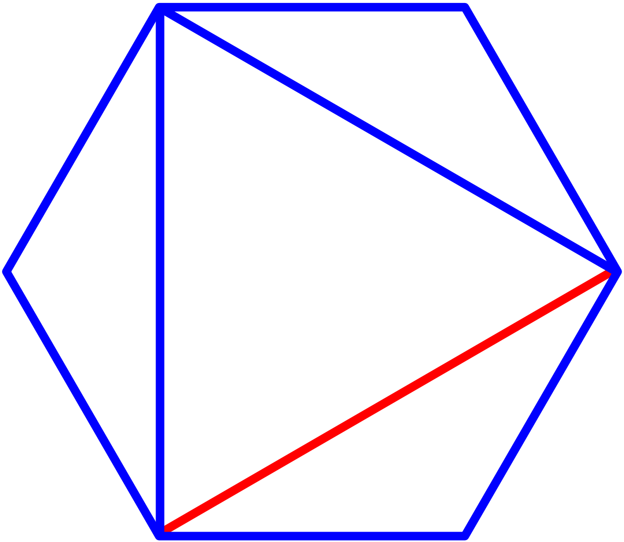

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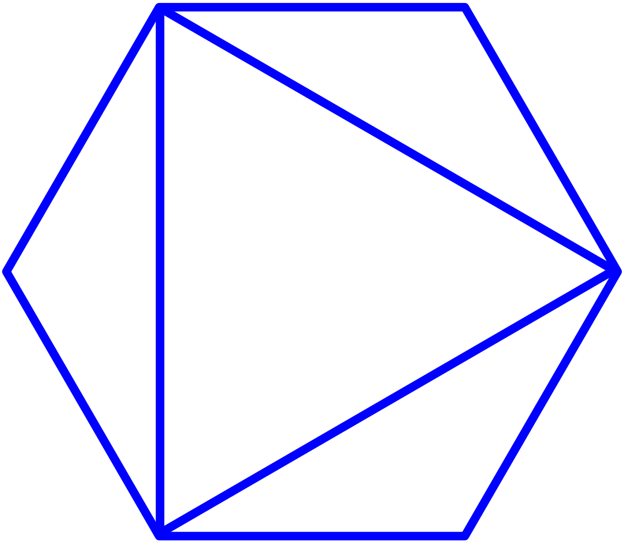

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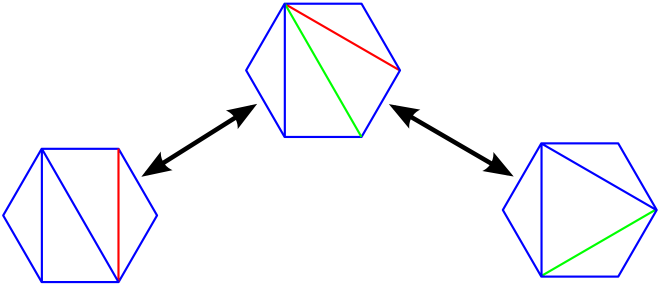

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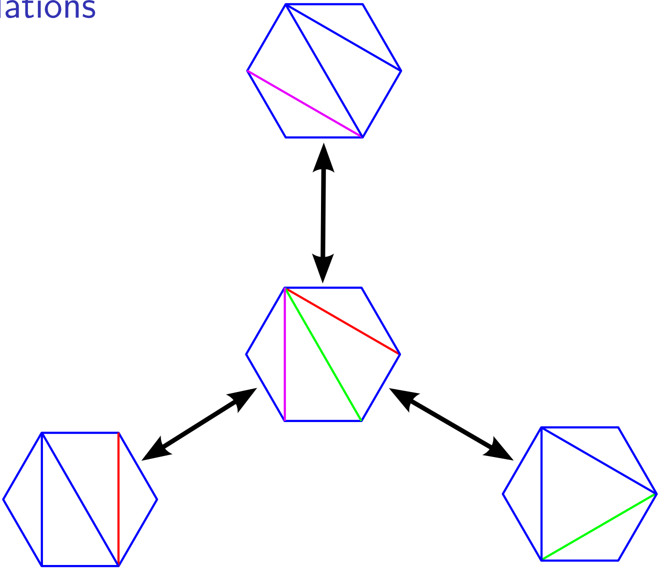

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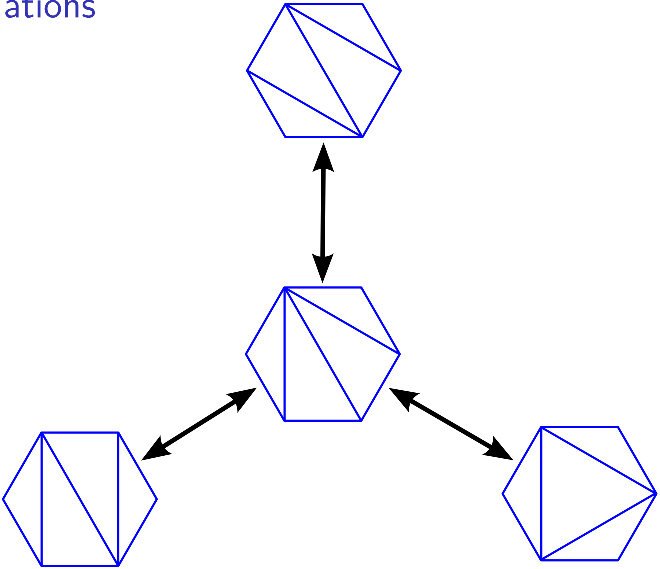

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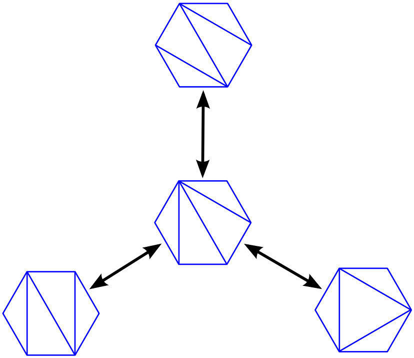

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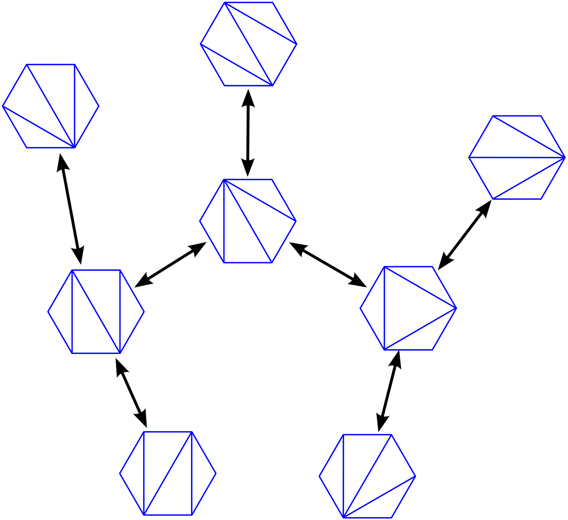

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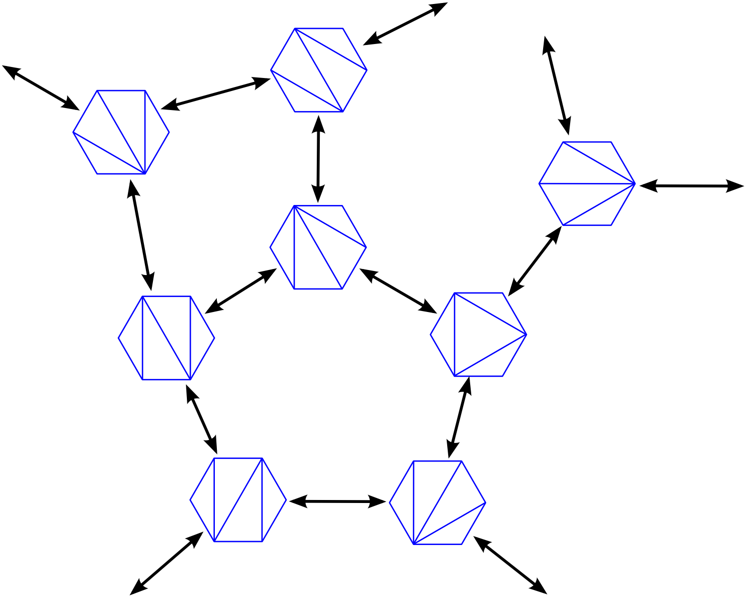
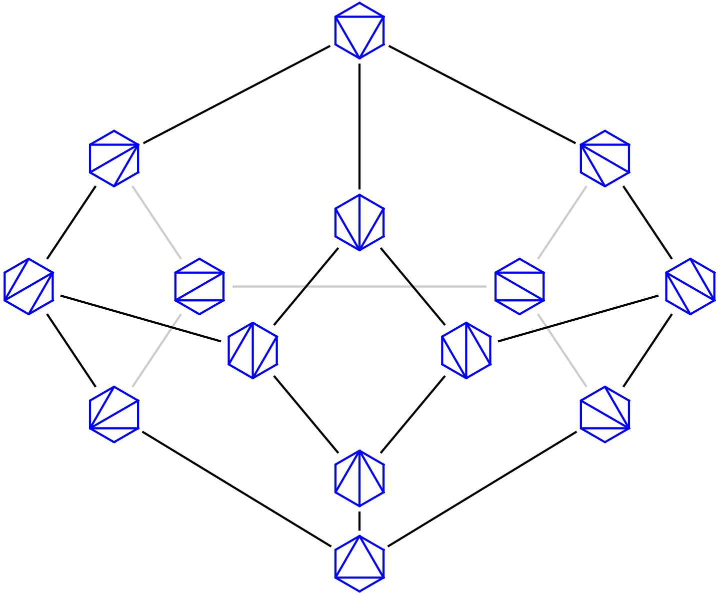
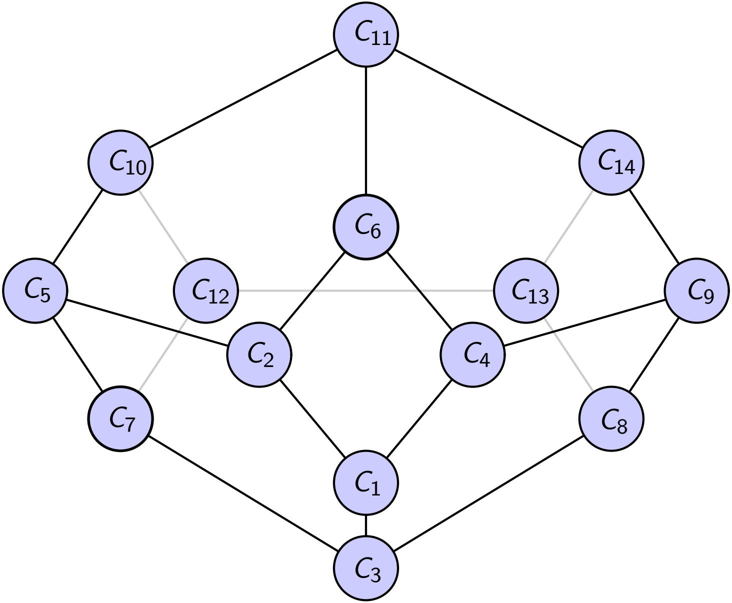


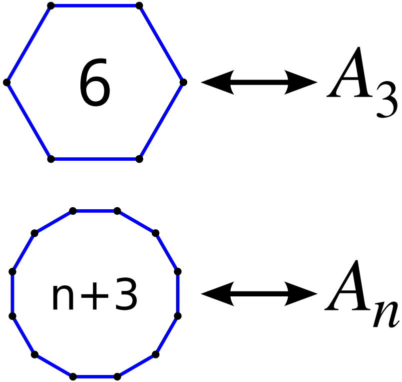
The Hexagon Triangulation Graph



The A_3 Exchange Graph



Triangulations and A_n



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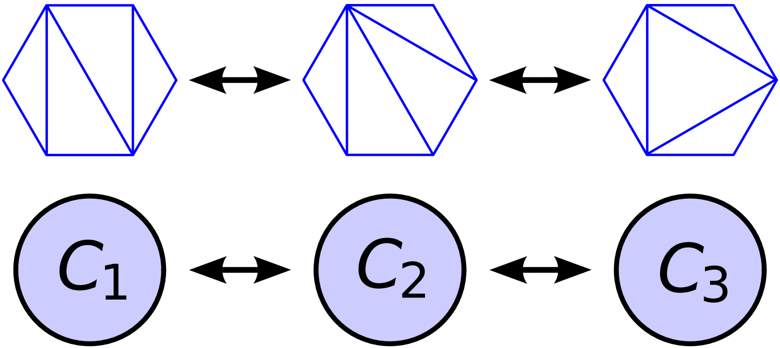

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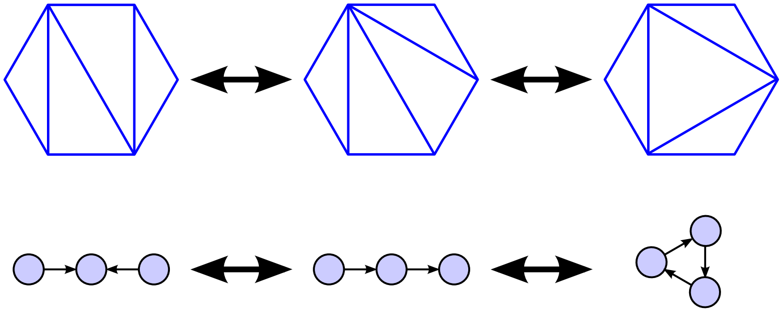

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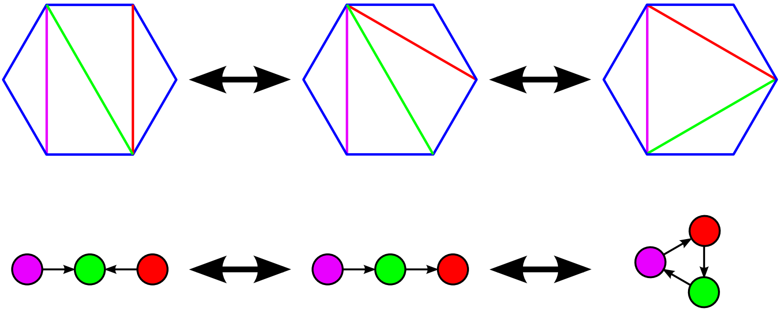
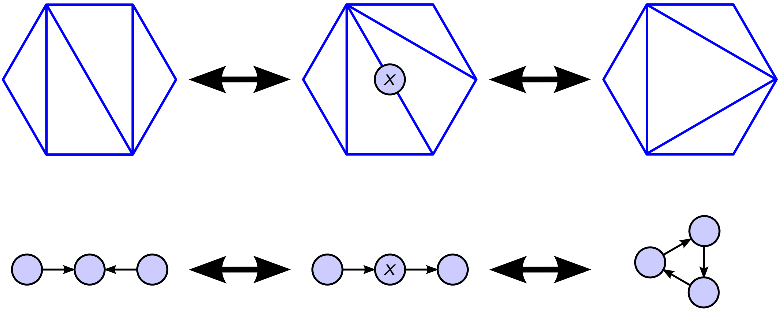


Triangulations and Cluster Variables



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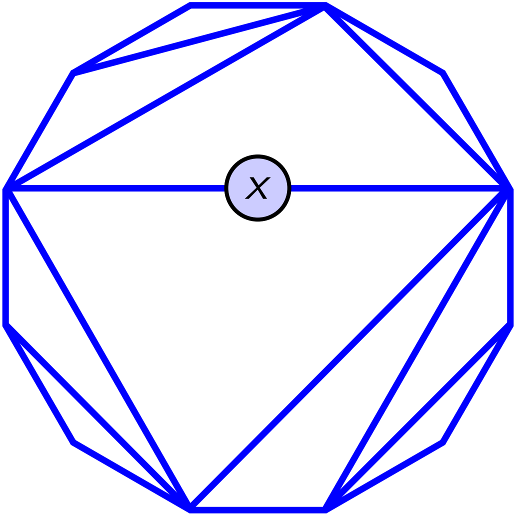

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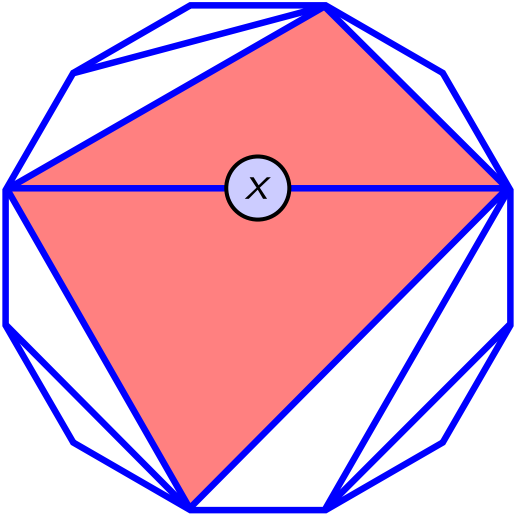

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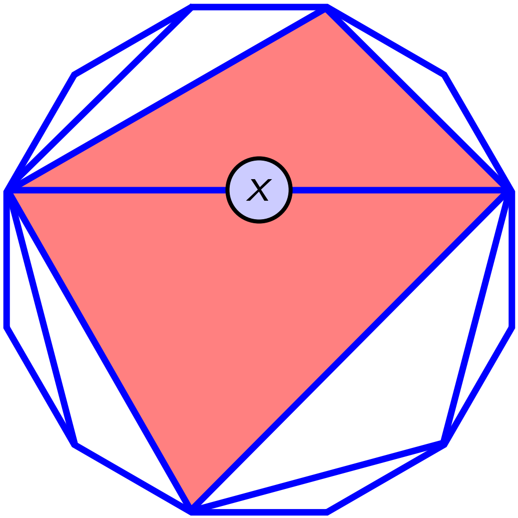

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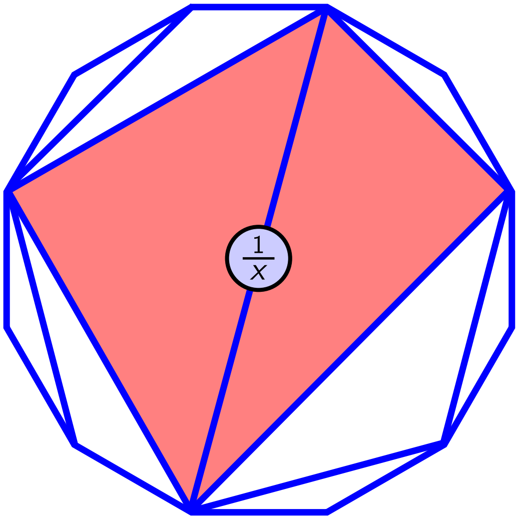

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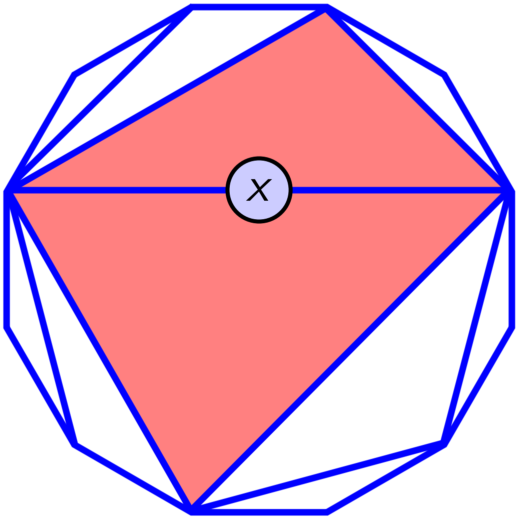

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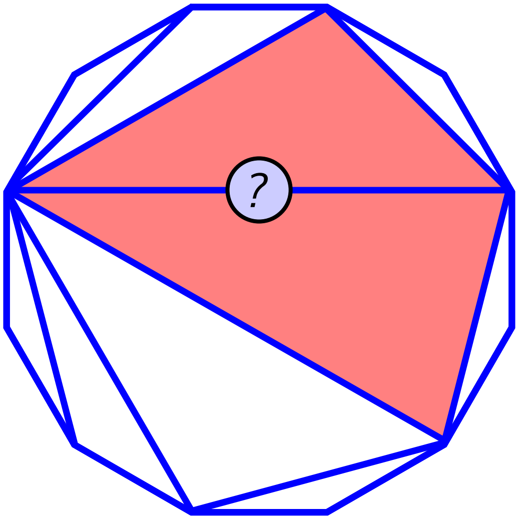
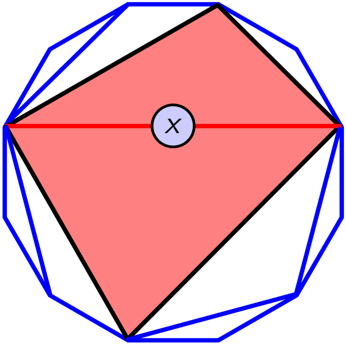
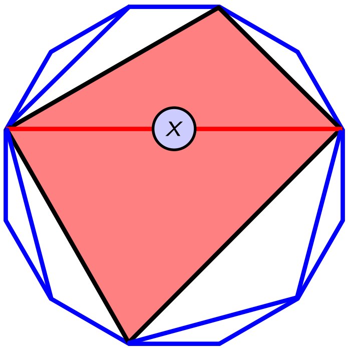


Triangulations and Cluster Variables – Recap

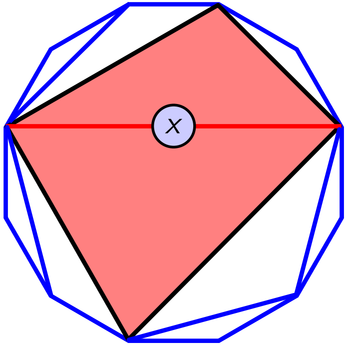


Triangulations and Cluster Variables – Recap



- Flip x chord: $x \mapsto 1/x$

Triangulations and Cluster Variables – Recap



- Flip x chord: $x \mapsto 1/x$
- Flip nonadjacent chord: $x \mapsto x$

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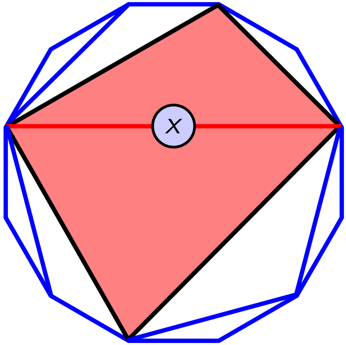

- Flip x chord: $x \mapsto 1/x$
- Flip nonadjacent chord: $x \mapsto x$
- Flip adjacent chord: $x \mapsto ?$

Brown's Theorem Redux

Theorem (Brown): If there is a clique of size n in the A_n cluster algebra, then there exists a minimal algebraic generating set for the space of cluster polylogarithms $\mathcal{A}$.

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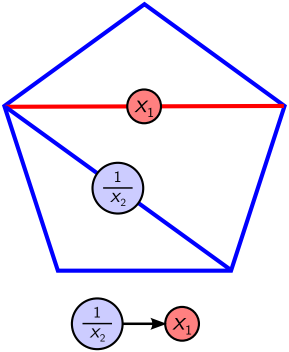
Cliques on A_2



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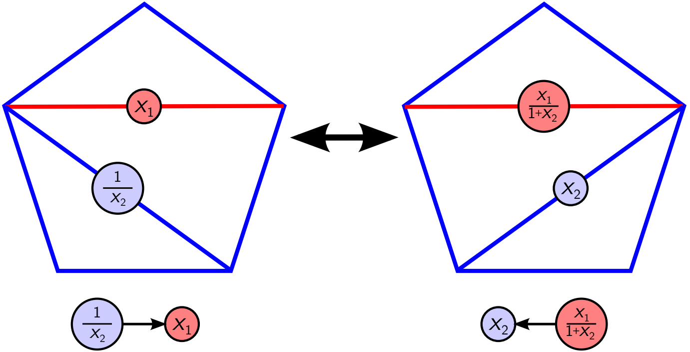

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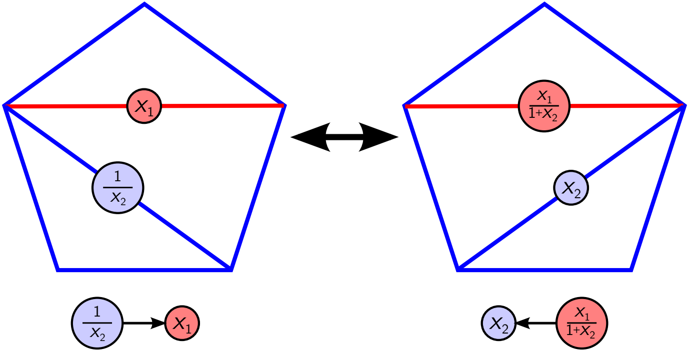
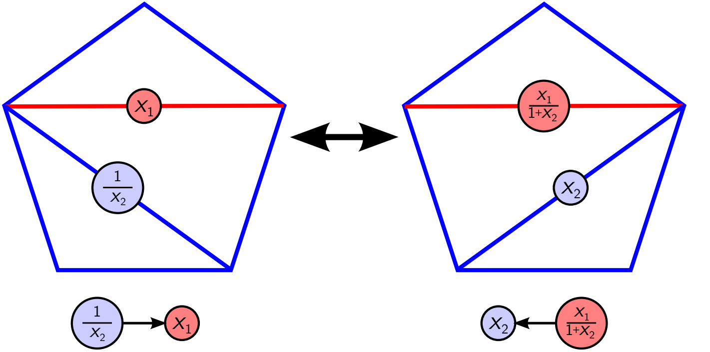


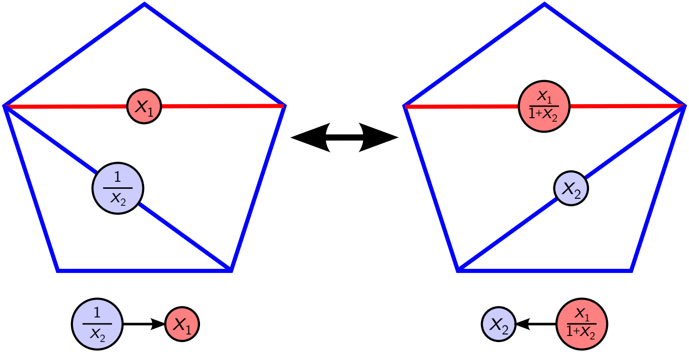
$$\boxed{x_1} - \boxed{\frac{x_1}{1+x_2}} = ?$$

Cliques on A_2



$$\boxed{x_1} - \boxed{\frac{x_1}{1+x_2}} = \frac{x_1(1+x_2) - x_1}{1+x_2} = \frac{x_1 x_2}{1+x_2} = \boxed{x_2} \times \boxed{\frac{x_1}{1+x_2}}$$

Cliques on A_2



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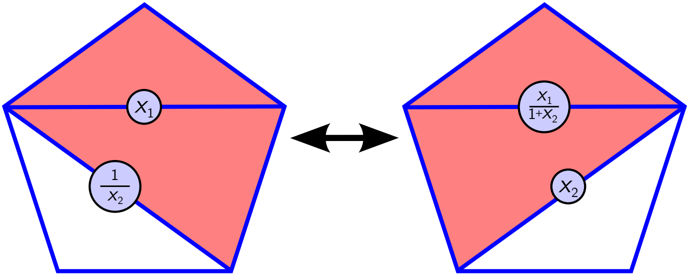
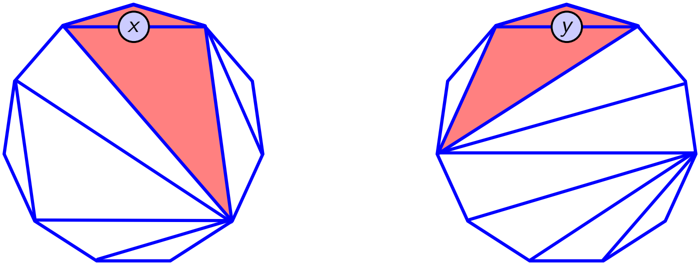
$$\left\{ x_1, \frac{x_1}{1+x_2} \right\} \text{ is a 2-clique of } A_2$$

Cliques on A_2



Cliques on A_n



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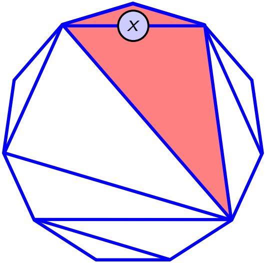

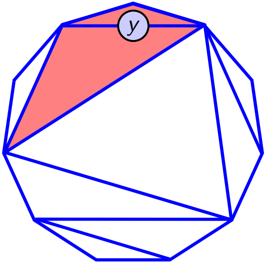

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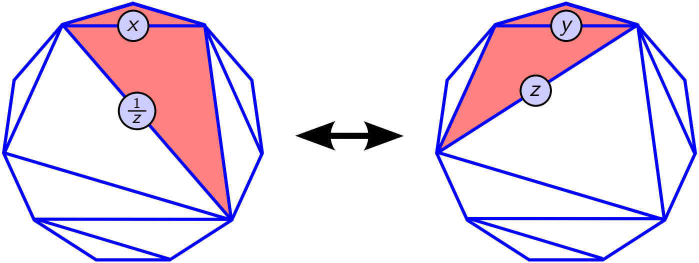

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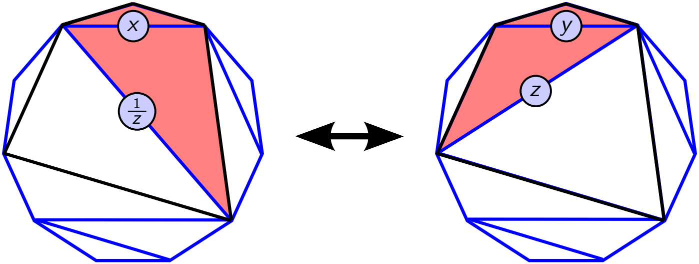

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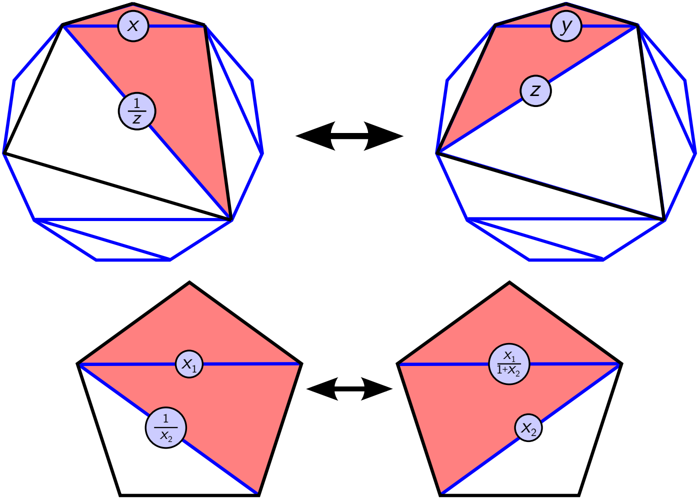

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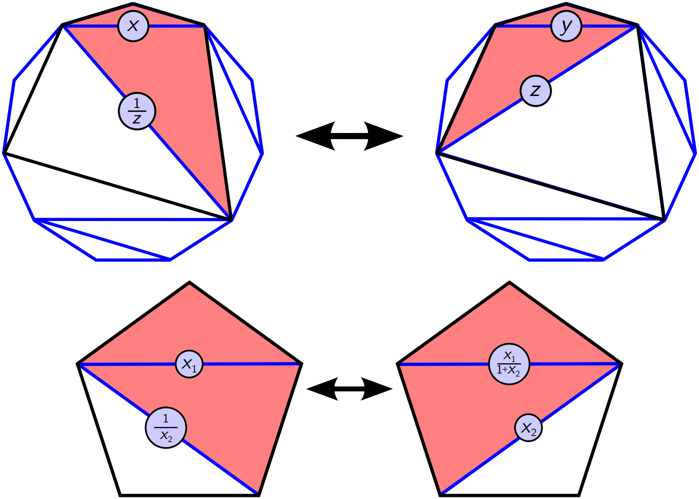
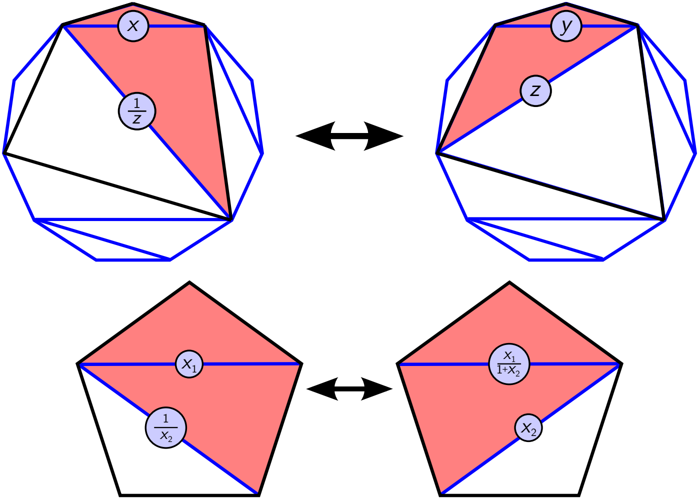


$$\boxed{x} - \boxed{y} = \boxed{z} \times \boxed{y}$$

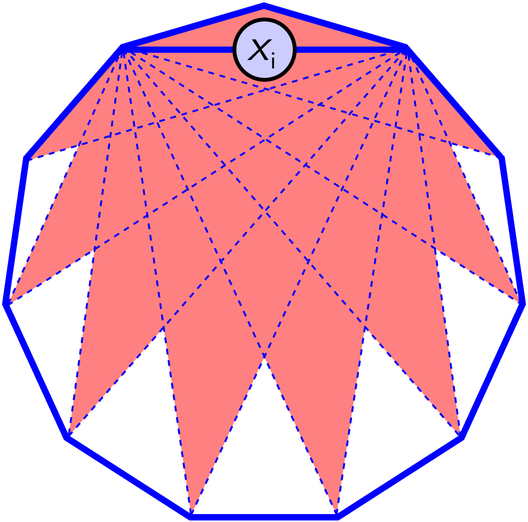
Cliques on A_n



$$\boxed{x} - \boxed{y} = \boxed{z} \times \boxed{y}$$

$$\left\{\ldots, \boxed{x}, \ldots ? \ldots, \boxed{y}, \ldots\right\}$$

The Hedgehog Theorem: n-Cliques on A_n



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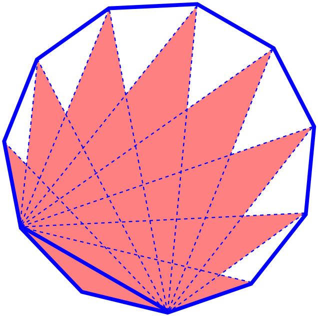

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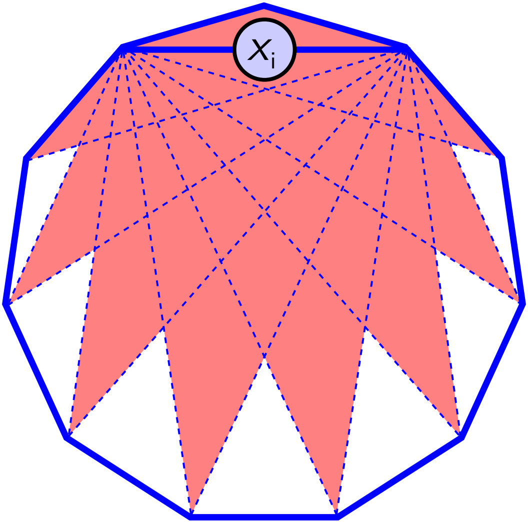
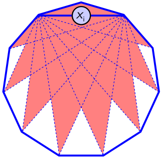


$\left\{ \boxed{x_1}, \boxed{x_2}, \ldots, \boxed{x_n} \right\}$ is an *n*-clique on A_n!

Recap

Clique

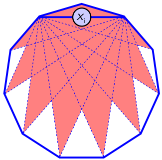


Recap

Clique

↓

Hedgehog Basis for A_n



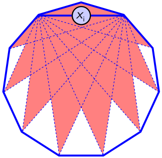
$$\bigcup_{k \in \mathbb{K}} \bigcup_{i=1}^{n} G_k \left[\{ A_n \text{ clique} \} \right]$$

Recap

Clique

↓

Hedgehog Basis for A_n



↓

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Cluster Polylogarithms

$$\mathcal{A}_n = \left\{ \mathcal{A} : \mathbf{Gr}(4,n)/\left((\mathbb{C}^*)^{n-1}\right) \xrightarrow{\text{cluster algebra}} \mathbb{C}^n \xrightarrow{\text{polylog}} \mathbb{C} \right\}$$

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Clique

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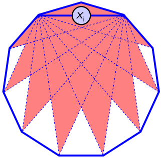

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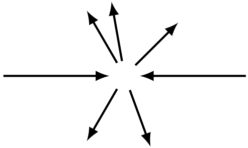
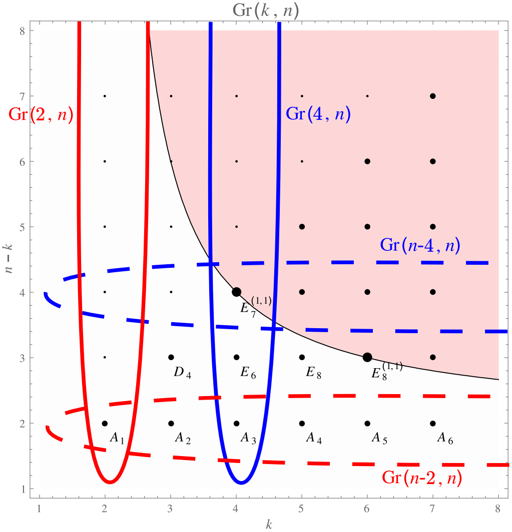
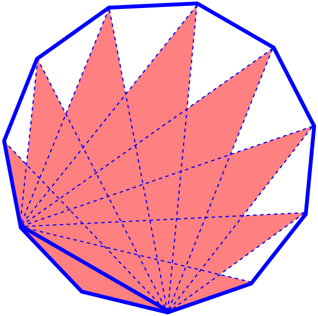
Scattering Amplitudes



Future Research



Thank You!



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